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Liquidity and Ambiguity: Banks or Asset Markets?*

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Abstract

This paper studies the impact of incalculable risk (i.e. ambiguity) on two alternative institutional arrangements for financial intermediation in an economy where consumers face uncertain liquidity needs. The ambiguity the consumers experience is modeled by their degree of confidence in their additive beliefs. The optimal liquidity allocation and two institutional arrangements for implementing this allocation are analyzed: a secondary asset market and a competitive banking sector.

For full confidence we obtain the well-known result that consumers prefer the deposit contract offered in the competitive banking sector over the asset market, since the former can provide the optimal cross subsidy for consumers with high liquidity needs. With increasing ambiguity this preference will be reversed: the asset market is preferred, since it avoids inefficient liquidation if the bank reserve holdings turn out to be suboptimal.

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Keywords: Financial institutions, Liquidity, Incalculable risk, Ambiguity, Choquet Expected Utility.

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1 Introduction

Many economic activities depend crucially on facilities enabling economic agents to raise liquid funds against claims on their future income. Future income streams and liquidity needs are by nature uncertain and, therefore, difficult to contract on. Hence, we find a plethora of financial institutions serving the purpose of providing liquidity, ranging from credit contracts, which allow customers to raise liquid funds against claims on future payments, to secondary asset markets, in which organized trade of illiquid assets can take place. In the case of credit contracts, loans require sufficient collateral or a more or less complicated assessment procedure of future payments. Secondary asset markets can be established for homogeneous asset categories with sufficiently regular demand and supply that justify the costs of an organized market. Bank deposit contracts are a special instrument of liquidity provision. Banks accept deposits of liquid funds and promise to repay liquid funds in the future at any time the depositor claims them back. Deposits offer banks the opportunity to invest at least part of these funds in illiquid assets, since normally only a fraction of deposits will be called upon in any period.

In this paper, we contrast two institutional arrangements for liquidity provision: a *bank deposit contract* as in Diamond and Dybvig (1983) and a secondary *asset market* as in Diamond (1997) and Allen and Gale (2004). Difficulties with liquidity provision arise from the inherent uncertainty about asset returns, but also from uncertainty about individual and aggregate liquidity needs. We will focus here on uncertainty about individual and aggregate liquidity needs and will disregard uncertainty about asset returns[1].

In most of the existing economic literature, uncertainty is viewed as ignorance about the outcome of a random draw from a known probability distribution, where the known probability distribution is identified with the actual frequency of these outcomes in a population. For example, consumers assume that preference parameters, which determine their private liquidity needs in the future, are randomly drawn with probabilities equal to the actual frequencies of these preference parameters in the population.

We will show that ambiguity about these distributions can change well-known results of the literature on financial intermediation. In particular, we will argue that institutional arrangements, such as bank deposit contracts and secondary asset markets, are quite distinct in their robustness with respect to investors' ambiguity about individual and aggregate liquidity needs[2].

[1] The impact of uncertain asset returns is addressed in Jacklin and Bhattacharya (1988).

[2] For studies on the impact of ambiguity in the context of financial markets and monetary policy see e.g. Ford *et al.* (2006), Spanjers (2008a), Spanjers (2008b), and Spanjers (2018).

1.1 Modelling ambiguity

Uncertainty has long been recognized as an important factor determining economic activities. The distinction between risk, i.e., situations where the probabilities of events are known, and ambiguity, i.e., situations where this is not the case, has served Knight (1921) as the foremost explanation of economic phenomena such as profit and entrepreneurial activity.

For several decades the behaviorist theory of subjective expected utility by Savage (1954) appeared to have rendered this distinction obsolete. If individuals faced with uncertainty behave as if they held a subjective probability distribution over events, then, from an analytical point of view, behavior under risk and under uncertainty can be treated in the same way. Yet early evidence by Ellsberg (1961) suggested that the hypothesis of a well-defined subjective probability distribution cannot be maintained empirically. Systematic laboratory experiments have confirmed Ellsberg's conjecture, see Camerer and Weber (1992). It appears to be well-established now that certain aspects of uncertainty cannot be captured by the assumption of a subjective probability distribution.

In recent years, substantial progress has been made in modelling decision-making under uncertainty without subjective probabilities. Schmeidler (1989) and Gilboa (1987) proposed a theory where decision makers' beliefs are represented by non-additive probabilities (or capacities). Choquet expected utility (CEU) theory, a generalization of subjective expected utility, can accommodate many different weighting schemes for events while maintaining some separation of beliefs and outcome evaluation, which is important in economic applications in order to identify risk preferences.

Generalizing additive beliefs to non-additive beliefs allows one to accommodate empirically observed anomalies like, e.g., the Ellsberg-paradox[3]. Without imposing additional restrictions on capacities, however, predictions about economic behavior are typically less precise. In this paper, we will restrict attention to beliefs that can be represented by simple capacities. Simple capacities are a special case of E-capacities which are studied in Eichberger and Kelsey (1999)) and of NEO-additive capacities which are analyzed and axiomatized in Chateauneuf *et al.* (2007).

With a simple capacity, the Choquet expected utility of an act is a weighted average of the expected utility of this act with respect to an additive probability distribution and the minimum utility obtainable with the act. The weight attached to the expected utility part can be interpreted as the degree of confidence of the decision maker in the additive probability. Hence, one can view ambiguity as lack of confidence in an additive belief.

[3]See Ellsberg (1961).

1.2 Liquidity and financial intermediation

According to Diamond and Dybvig's seminal article, liquidity problems arise because consumers, who do not know their private liquidity needs, have to decide on investments which require a long-term commitment of their funds. Since liquidity needs are private information, direct contracting is impossible and an agency problem arises. In this context, one can raise and answer important questions about the institutional design of financial intermediation and its regulation.

The earlier literature assumes that the illiquid asset can be liquidated at par.[4] With this assumption, there is no liquidity problem but an incompleteness of contract due to unrealized insurance opportunities which can be accommodated by a bank deposit contract. The more recent literature assumes that the illiquid asset cannot be liquidated at all. Hence, a combined problem of insufficient liquidity and incomplete insurance contracts arises[5]. Diamond (1997) shows that secondary asset markets can resolve the liquidity problem but cannot provide the necessary cross subsidy in order to deal with the incomplete insurance issue. In contrast, a bank deposit contract can solve both problems and implement the optimal contract. A central question in this literature concerns the precarious coexistence of banks and secondary asset markets. Diamond (1997) and Allen and Gale (2004) assume restricted access of consumers to the secondary asset market. With restricted market participation consumers can secure liquidity and obtain, at least partially, the cross subsidy required by the optimal contract.

In this paper, we will assume that the illiquid asset can be liquidated at some cost. Compared to the literature, this is an intermediate case. There is a liquidity problem, but a perfect commitment not to liquidate early is impossible. In order to focus on ambiguity effects we assume that consumers have risk neutral preferences, as in Chari and Jagannathan (1988). Nevertheless, a cross subsidy problem occurs if some consumers' return from holding the liquid asset exceeds the return on the illiquid asset. The ex-ante optimal contract requires a cross subsidy from consumers with low liquidity needs to consumers with high liquidity needs, though not for insurance reasons. Hence, without ambiguity, we obtain the results of Diamond (1997).

The main contribution of this paper consists in the analysis of the role of ambiguity for intermediary institutions. Comparing bank deposits and secondary asset markets, we obtain the following results. In the presence of ambiguity, neither bank deposit contracts nor the asset market implement the ex-ante optimal allocation of liquidity. The evaluation of institutions depends on the degree of ambiguity which consumers experience regarding the probability distribution over private and aggregate liquidity needs. With high ambiguity, neither bank deposits nor asset trading can improve upon

[4]See Diamond and Dybvig (1983) and Jacklin (1987).

[5]See Diamond (1997) and Allen and Gale (2004).

the allocation without intermediation. For middle levels of ambiguity, a secondary asset market is the preferred institution. If the level of ambiguity is low, bank deposit contracts offer the ex-ante preferred method of liquidity provision.

1.3 Organization of the paper

In the following section describes the economic model, analyzes individual behavior without intermediation, and studies the optimal incentive-compatible allocation. The following two sections deal with the secondary asset market and the bank deposit contract, respectively. Section 5 compares the performance of these institutions under ambiguity, Section 6 contains concluding remarks. In Appendix A, we embed the decision criteria used in this paper in the more general theory of capacities and the Choquet integral. Longer proofs are gathered in Appendix B.

2 The economy

We consider an economy with many, relative to the market, small consumers. This is modelled by the assumption of a continuum set of consumers, the interval $[0, 1]$. The economy extends over three periods. In Period 0, each consumer is endowed with one unit of wealth (money) and faces the following investment opportunities:

Table 1: Asset payoffs

		Payoff in		
	Assets	Period 0	Period 1	Period 2
1.	Asset matured	-1	0	α_2
	Asset liquidated	-1	α_1	0
2.	Money 0 to 1	-1	1	0
	Money 1 to 2	0	-1	1

We assume $\alpha_2 > 1 > \alpha_1$. This payoff structure justifies calling the asset illiquid. The asset offers a long-term investment possibility with a better return than money, if held to maturity in Period 2. Compared to money it is illiquid, however, since the liquidation payoff α_1 in Period 1 falls short of the return from holding money.

Remark 1. Uncertainty about liquidity needs is the focus of this paper. The economy model is similar to the one in Diamond and Dybvig (1983) and Diamond (1997). As in these articles we do not assume that the payoffs of the assets are uncertain. The restriction to certain payoffs of the asset is not essential[6]. Moreover, it maintains comparability with the results in this literature.

There are two types of ex-ante identical consumers. In Period 1, consumers privately learn their type $t \in \{h, \ell\}$. The type of a consumer determines his preference for liquidity in Period 1. Type-dependent preferences are represented by a risk-neutral von Neumann-Morgenstern utility index

$$u(z_1, z_2; t) = \beta_t \cdot z_1 + z_2$$

where z_1 and z_2 denote consumption in Period 1 and Period 2, respectively. Throughout the paper, the following assumption about the parameter values of our model is maintained.

Assumption 1. *Liquidity preference*

$$\textit{(i)} \qquad \beta_h > \alpha_2 > \beta_\ell \geq 1,$$

$$\textit{(ii)} \qquad \alpha_2 > \alpha_1 \cdot \beta_h.$$

According to Assumption 1 (i), consumers of type h strictly prefer to hold money, while consumers of type ℓ prefer an investment in the illiquid asset. Assumption 1 (ii) guarantees that the liquidation value of the illiquid asset α_1 is so low that it does not pay for a consumer with high liquidity needs to liquidate an investment in the illiquid asset[7] in Period 1.
The liquidation value $\alpha_1 \in (0, 1)$ falls between two extreme cases. In Diamond and Dybvig (1983) the long-term asset has the same liquidity as money, $\alpha_1 = 1$, while in Diamond (1997) it has no payoff in Period 1, $\alpha_1 = 0$. Here we assume some illiquidity $\alpha_1 < 1$, for, otherwise, the payoffs of the illiquid asset would simply dominate holding money[8].

Remark 2. In contrast to Diamond and Dybvig (1983), we do not assume a concave utility over consumption. The essence of our argument does not rely on the insurance against random liquidity needs which are studied in Diamond and Dybvig (1983). Here, liquidity needs arise from the time structure of payoffs, the liquidity costs α_1 and preferences for early consumption in a section of the population. With this respect our model is similar to Chari and Jagannathan (1988). Because we abstract from risk aversion and

[6]Spanjers (1999, Chapter 5) extends the comparison between banks and asset markets to the case of ambiguity about the illiquid asset's return.

[7]This part of Assumption 1 is not strictly necessary for our analysis. It is however useful for the exposition since it allows us to skip discussing several cases which are of little interest.

[8]For $\alpha_1 = 0$, as in Diamond (1997), liquidation is impossible. Hence, any scheme which collects funds in Period 0 and redistributes them in Period 1 can perfectly commit to not liquidating early in favor of early consumers at the expense of late customers. This assumption will be important below, when we consider the deposit contract, and will be discussed in more detail there.

insurance considerations, our results can be unambiguously traced to the uncertainty aversion of consumers, implicit in the Choquet expected utility, and the varying degree of ambiguity.

Uncertainty of consumers concerns their own liquidity needs and the aggregate demand for liquidity. Simple capacities, which model ambiguity here, are based on an additive probability distribution over these variables. As in Diamond and Dybvig (1983), it is assumed that the probability of a consumer being assigned a particular type equals the proportion of this type in the economy. In addition, we will assume that there is also uncertainty about the proportion of consumers of either type. Hence, in Period 0, both individual and aggregate liquidity needs are unknown.

Denote by τ the unknown proportion of consumers with high liquidity needs h. In Period 0, consumers know neither their type t nor the proportion τ of h-types in the population. While the proportion of consumers with high liquidity needs τ becomes common knowledge in Period 1, consumers learn only privately the information about their type t.

Beliefs of consumers are represented by a subjective joint probability distribution P over the unknown parameters $(t, \tau) \in \{h, \ell\} \times [0, 1]$. Ambiguity is modelled as lack of confidence γ in this additive probability distribution P. The following assumption characterizes the additive probability distribution P.

Assumption 2. Beliefs

1. Population shares:
   Conditional on the population share τ the probability distribution over types t equals the proportions of types in the economy, whenever this conditional probability is well-defined,

$$P(h|\tau) = \tau \qquad \textit{and} \qquad P(\ell|\tau) = 1 - \tau.$$

2. Correct beliefs:
   Consumers' marginal beliefs about the population share of h-types are concentrated on the true proportion π,

$$p(\tau) = \begin{cases} 1 & \textit{for} \quad \tau \geq \pi \\ 0 & \textit{otherwise} \end{cases},$$

   where $p(\tau)$ denotes a cumulative distribution function on the set of population shares for the h-types, $[0, 1]$.

In order to make our results comparable with the literature, e.g. with Diamond and Dybvig (1983), Jacklin (1987), Diamond (1997) and Allen and Gale (2004), beliefs about the proportion of consumers with high liquidity needs are identified with their actual population share π. Though consumers

have point predictions for the population shares of each type, which will turn out to be correct in Period 1, they may still experience ambiguity about these predictions in Period 0.
Assumption 2 implies the following joint probability distribution:

$$P(h,\tau) = \begin{cases} \pi & \text{if} \quad \tau = \pi \\ \\ 0 & \text{otherwise} \end{cases}, \quad P(\ell,\tau) = \begin{cases} 1-\pi & \text{if} \quad \tau = \pi \\ \\ 0 & \text{otherwise} \end{cases}.$$

Remark 3. Though there is an additive probability distribution representing the focal beliefs of consumers in Period 0 this does not imply that there is no uncertainty. Uncertainty is introduced by the degree of confidence γ in this focal belief as discussed in Appendix A. Modelling ambiguity as a lack of confidence allows us to view the reference case without ambiguity as the special case of $\gamma = 1$.

2.1 Investment without intermediation

If there are no intermediary institutions, then each consumer simply decides on the fraction of wealth m to be held as money and on the fraction to be invested in the illiquid asset, $1 - m$. This decision yields consumption $(z_1, z_2) = (m, \alpha_2 \cdot (1-m))$ and, for type t, the utility $u(m, \alpha_2 \cdot (1-m); t) = \beta_t \cdot m + \alpha_2 \cdot (1-m)$. By Assumption 1 (ii), $\alpha_2 > \alpha_1 \cdot \beta_h$, we need not consider the case of consumers wanting to liquidate their long-term investment.
In Period 0, when consumers have to choose their investment strategy m, they are uncertain both about their type t and the proportion of types in the population, τ. If the proportion of type h consumers τ were known, the ex-ante expected utility of a consumer would be $[\tau \cdot \beta_h + (1-\tau) \cdot \beta_\ell] \cdot m + \alpha_2 \cdot (1-m)$. If consumers lack confidence in the probability distribution $P(t,\tau)$, i.e., $\gamma < 1$, then the ex-ante Choquet expected utility of a consumer is

$$\begin{aligned} CEU(m;\gamma) = \quad & \gamma \cdot [(\pi \cdot \beta_h + (1-\pi) \cdot \beta_\ell) \cdot m + \alpha_2 \cdot (1-m)] \\ & +(1-\gamma) \cdot \min_{(t,\tau) \in \{h,\ell\} \times [0,1]} [\beta_t \cdot m + \alpha_2 \cdot (1-m)] \\ & = [\gamma \cdot \pi \cdot \beta_h + (1 - \gamma \cdot \pi) \cdot \beta_\ell] \cdot m + \alpha_2 \cdot (1-m). \end{aligned}$$

Maximizing $CEU(m;\gamma)$ over $m \in [0,1]$ yields the maximal ex-ante Choquet expected utility as a function of the degree of confidence γ, a result which we summarize in Proposition 1.[9]

Proposition 1. *In an economy without intermediation, the optimal invest-*

[9] We will use the index n in order to distinguish optimal decisions if there is **n**o intermediation.

ment policy

$$m_n^*(\gamma) = \begin{cases} 0 & \textit{for} \quad \gamma \cdot \pi < \frac{\alpha_2 - \beta_\ell}{\beta_h - \beta_\ell} \\ \in [0,1] & \textit{for} \quad \gamma \cdot \pi = \frac{\alpha_2 - \beta_\ell}{\beta_h - \beta_\ell} \\ 1 & \textit{for} \quad \gamma \cdot \pi > \frac{\alpha_2 - \beta_\ell}{\beta_h - \beta_\ell} \end{cases},$$

yields the maximal ex-ante Choquet expected utility

$$V_n^*(\gamma) = CEU_n(m_n^*(\gamma); \gamma) = \max\{\alpha_2\,,\, \gamma \cdot \pi \cdot \beta_h + (1 - \gamma \cdot \pi) \cdot \beta_\ell\}.$$

Consumers of type h prefer to hold money in Period 1. Hence, money holding is the optimal investment policy if both the assessed probability of becoming a consumer with high liquidity needs, π, and the degree of confidence in this belief, γ, are sufficiently high. Otherwise, all money is invested in the illiquid asset.

Since the illiquid asset has a certain return α_2, it is the preferred choice if the low type β_ℓ is realized. Hence, it becomes the default option for consumers with a high degree of ambiguity. Clearly, money would become the default option for low confidence if the illiquid asset had an uncertain return.

2.2 Optimal contract

The allocation which consumers can generate individually in this economy is suboptimal. Even taking into account the informational constraints, there is scope for Pareto improvements by pooling resources in Period 0 and investing them jointly. For example, suppose there was no ambiguity, $\gamma = 1$, then one could pool consumers' funds in Period 0, invest the proportion $(1 - \pi)$ in the illiquid asset and hold the remaining proportion π as money. This investment strategy would allow to pay out one unit of money to consumers with high liquidity need in Period 1 and the amount α_2 to consumers with low liquidity needs in Period 2. Although not optimal, this payout would yield an expected utility exceeding the expected utility which consumers can guarantee themselves,

$$\pi \cdot \beta_h + (1 - \pi) \cdot \alpha_2 > V_n^*(1).$$

Moreover, such a payout scheme would be incentive-compatible, since consumers with low liquidity need β_ℓ would not claim a payout in Period 1 ($\beta_\ell < \alpha_2$). As Diamond (1997) also points out a pure liquidity provision scheme, although improving upon autarky, is not optimal. Some cross subsidy between consumer types is necessary for an optimal contract. Hence, an optimal asset pooling scheme needs to be studied carefully. The optimal incentive-compatible contract will be derived in this subsection.

For an optimal allocation, the resources of all consumers are pooled in Period 0 and optimally invested in money and the illiquid asset. In Period 0, before types $t \in \{h, \ell\}$ are *privately known* and before the proportion of types τ is *common knowledge*, all consumers are identical. In Period 1 all uncertainty is resolved, however, consumers' types are not publicly known. Hence, the optimal contract will assign type-contingent payouts, $z_h = (z_{1h}, z_{2h})$ and $z_\ell = (z_{1\ell}, z_{2\ell})$, subject to self-selection constraints, which reflect the private information of consumers. Moreover, an optimal allocation must maximize individual utility without ignoring the informational asymmetry.

In Period 1, the optimal type-contingent payout scheme (z_h, z_ℓ) will maximize the *average utility* of consumers given the, then known, public information about the population share. The *average utility* for a population with a fraction τ of $h-$type consumers is

$$\begin{aligned} U(z_h, z_\ell; \tau) = \;& \tau \cdot u(z_h; h) + (1-\tau) \cdot u(z_\ell; \ell) \\ = \;& \tau \cdot [\beta_h \cdot z_{1h} + z_{2h}] + (1-\tau) \cdot [\beta_\ell \cdot z_{1\ell} + z_{2\ell}]. \end{aligned} \tag{1}$$

The choice problem of the optimal contract has two stages. In Period 0, the fraction M of aggregate wealth held as money is determined. The remaining wealth is invested in the illiquid asset. In Period 1, once the investment decision M has been taken and once the proportion of $h-$type consumers, $\tau \in [0, 1]$, has become public knowledge, the type-contingent payouts (z_h, z_ℓ) are determined. We analyze these two stages in turn.

2.2.1 The optimal payout scheme

Given the aggregate money holdings M and a realized proportion τ of $h-$type consumers, the optimal payout scheme (z_h, z_ℓ) must maximize consumers' average utility subject to self-selection and feasibility constraints:

$$\begin{array}{lllll} \max\limits_{z_h, z_\ell} & U(z_h, z_\ell; \tau) & & & \\ \text{s.t.} & \beta_h \cdot z_{1h} + z_{2h} & \geq & \beta_h \cdot z_{1\ell} + z_{2\ell}, & S_h \\ & \beta_\ell \cdot z_{1\ell} + z_{2\ell} & \geq & \beta_\ell \cdot z_{1h} + z_{2h}, & S_\ell \\ & \tau \cdot z_{1h} + (1-\tau) \cdot z_{1\ell} & = & M, & F_1 \\ & \tau \cdot z_{2h} + (1-\tau) \cdot z_{2\ell} & = & \alpha_2 \cdot (1-M), & F_2 \\ & z_{1h} \geq 0,\; z_{1\ell} \geq 0,\; z_{2h} \geq 0,\; z_{2\ell} \geq 0. & & & \end{array} \tag{2}$$

The feasibility constraints, F_1 and F_2, guarantee that aggregate payouts in periods 1 and 2 can be financed given the investment policy M. According to Assumption 1 (ii), it can never be optimal to liquidate the long-term investment in Period 1, hence this possibility is disregarded in decision problem (2). Incentive-compatibility of the payout scheme follows from the two self selection constraints S_h and S_ℓ.

Disregarding the self-selection constraints S_h and S_ℓ, one optimal allocation would be $\widetilde{z}_{1\ell} = 0$, $\widetilde{z}_{1h} = \frac{M}{\tau}$, $\widetilde{z}_{2h} = 0$ and $\widetilde{z}_{2\ell} = \frac{\alpha_2 \cdot (1-M)}{1-\tau}$, yielding the optimal average utility

$$\mathcal{U}(M, \tau) = \beta_h \cdot M + \alpha_2 \cdot (1-M). \tag{3}$$

For given initial money holdings M, however, it depends on τ whether this solution satisfies the self-selection constraints. The constraint S_h will be binding for $\beta_h \cdot \frac{M}{\tau} < \frac{\alpha_2 \cdot (1-M)}{1-\tau}$, i.e., for

$$\tau > \frac{\beta_h \cdot M}{\beta_h \cdot M + \alpha_2 \cdot (1-M)} = \overline{\tau}(M).$$

The other incentive constraint S_ℓ will bind, if $\frac{\alpha_2 \cdot (1-M)}{1-\tau} < \beta_\ell \cdot \frac{M}{\tau}$ holds, i.e., if

$$\tau < \frac{\beta_\ell \cdot M}{\beta_\ell \cdot M + \alpha_2 \cdot (1-M)} = \underline{\tau}(M). \tag{4}$$

The optimal solution from Equation (3) is valid for all $\tau \in [\underline{\tau}(M), \overline{\tau}(M)]$.
From the linearity of the average utility function $U(z_h, z_\ell; \tau)$ in Equation (1) it is immediately clear that the optimal allocation fails to be unique at an optimum where the self-selection constraints are not binding. Any second-period consumption allocation $(z_{2h}, z_{2\ell})$ satisfying the self-selection constraints and the constraint $\tau \cdot z_{2h} + (1-\tau) \cdot z_{2\ell} = \alpha_2 \cdot (1-M)$ would also be optimal. It is therefore possible to transfer second-period consumption from ℓ-type to h-type consumers at no cost in terms of the average utility. Hence, the optimal average utility of Equation (3) remains unchanged for $\tau > \overline{\tau}(M)$.
In contrast, if the constraint S_ℓ is binding, then one has to decrease first-period consumption of consumers with high liquidity needs and increase first-period consumption of ℓ-type consumers in order to satisfy the self-selection constraints.
For $\tau < \underline{\tau}(M)$, one obtains the optimal allocation $\widetilde{z}_{1h} = M + \frac{\alpha_2}{\beta_\ell} \cdot (1-M)$, $\widetilde{z}_{1\ell} = M - \frac{\tau}{1-\tau} \cdot \frac{\alpha_2}{\beta_\ell} \cdot (1-M)$, $\widetilde{z}_{2h} = 0$ and $\widetilde{z}_{2\ell} = \frac{\alpha_2}{1-\tau} \cdot (1-M)$ yielding an average utility of

$$\begin{aligned}\mathcal{U}(M,\tau) &= \tau \cdot \left[\beta_h \cdot \left(M + \frac{\alpha_2 \cdot (1-M)}{\beta_\ell}\right)\right] \\ &\quad + (1-\tau) \cdot \left[\beta_\ell \cdot \left(M - \frac{\tau}{1-\tau} \cdot \frac{\alpha_2 \cdot (1-M)}{\beta_\ell}\right) + \frac{\alpha_2 \cdot (1-M)}{1-\tau}\right] \\ &= \frac{(\tau \cdot \beta_h + (1-\tau) \cdot \beta_\ell)}{\beta_\ell} \left[\beta_\ell \cdot M + \alpha_2 \cdot (1-M)\right].\end{aligned} \tag{5}$$

In summary, the maximal average utility obtainable with an optimal contract is a function of the population share of h-type consumers τ and the investment M made in Period 0:

$$\mathcal{U}(M,\tau) = \begin{cases} \frac{(\tau \cdot \beta_h + (1-\tau) \cdot \beta_\ell)}{\beta_\ell} \cdot [\beta_\ell \cdot M + \alpha_2 \cdot (1-M)] & \text{for} \quad \tau < \underline{\tau}(M) \\ \beta_h \cdot M + \alpha_2 \cdot (1-M) & \text{otherwise} \end{cases}. \tag{6}$$

2.2.2 The optimal investment policy

In Period 0, when consumers are still uncertain about their types and the proportion of types τ, one obtains the Choquet expected value of an investment decision M by taking the Choquet integral of the average utility $\mathcal{U}(M,\tau)$ in Equation (6). Notice that information about the types of consumers is not necessary for the optimal investment choice. This information becomes relevant in Period 1, when consumers privately know their types. The payout scheme derived in the previous subsection will guarantee truthful revelation of this information.

For any number $\varepsilon \in [0,1]$, denote by $\beta(\varepsilon) := \varepsilon \cdot \beta_h + (1-\varepsilon) \cdot \beta_\ell$ the expected return on money holdings in Period 1. By Assumption 2 the marginal probability distribution over population shares of the h-type consumers is concentrated on the true proportion π. Hence, $\mathcal{U}(M,\pi)$ is the expected average utility of a consumer in Period 0. The ex-ante worst case is obtained for the combination $t = \ell$ and $\tau = 0$ and yields a utility of $\mathcal{U}(M,0)$. Given a degree of confidence γ, the the ex-ante Choquet expected utility function is

$$CEU_o(M;\gamma) = \gamma \cdot \mathcal{U}(M,\pi) + (1-\gamma) \cdot \mathcal{U}(M,0) \tag{7}$$
$$= \begin{cases} \frac{\beta(\gamma\pi)}{\beta_\ell} \cdot [\beta_\ell \cdot M + \alpha_2 \cdot (1-M)] & \text{for} \quad \pi < \underline{\tau}(M) \\ \\ \beta(\gamma) \cdot M + \alpha_2 \cdot (1-M) & \text{otherwise} \end{cases} .$$

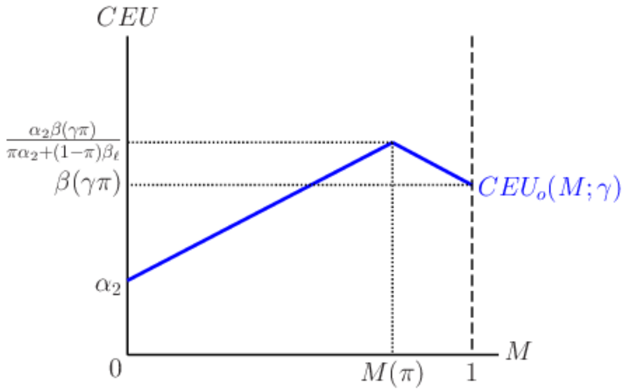
The optimal investment policy M will be chosen to maximize $CEU_o(M;\gamma)$ over all $M \in [0,1]$.

Condition $\pi < \underline{\tau}(M)$ in Equation (4) is equivalent to the condition

$$M > \frac{\pi \cdot \alpha_2}{\alpha_2 \cdot \pi + (1-\pi) \cdot \beta_\ell} = M(\pi).$$

Figure 1 shows the Choquet expected utility function of Equation (7).

Figure 1: Optimal reserves



It is immediately obvious that the optimal fraction of wealth held as money is

$$M_o^*(\gamma) = \begin{cases} M(\pi) & \text{for} \quad \beta(\gamma) > \alpha_2 \\ \in [0, M(\pi)] & \text{for} \quad \beta(\gamma) = \alpha_2 \\ 0 & \text{for} \quad \beta(\gamma) < \alpha_2 \end{cases} .$$

Substituting the optimal investment choice in the Choquet expected utility function, $CEU_o(M_o^*(\gamma); \gamma)$,
yields the optimal ex-ante Choquet expected utility as a function of the degree of confidence γ.

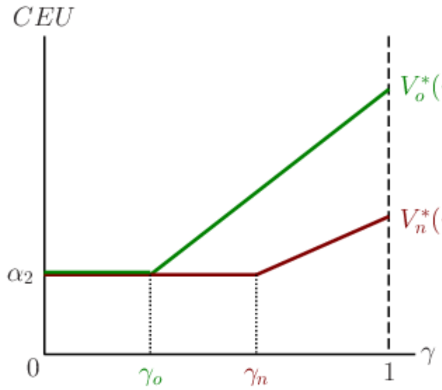
We summarize this result in the following proposition.

Proposition 2. *The maximal Choquet expected utility from an optimal contract is*

$$\begin{aligned} V_o^*(\gamma) &= CEU_o(M_o^*(\gamma); \gamma) \\ &= \alpha_2 \cdot \max\{, 1\} \\ &= \begin{cases} \frac{\alpha_2 \cdot \beta(\gamma \cdot \pi)}{\alpha_2 \cdot \pi + (1-\pi) \cdot \beta_\ell} & \text{if} \quad \beta(\gamma) > \alpha_2 \\ \\ \alpha_2 & \text{if} \quad \beta(\gamma) \leq \alpha_2 \end{cases} . \end{aligned}$$

The maximal ex-ante Choquet expected utility $V_n^*(\gamma)$ obtainable for a consumer in the absence of intermediary institutions was derived in Proposition 1. This value can be compared with the Choquet expected utility $V_o^*(\gamma)$ of an optimal contract from Proposition 2. Figure 2 shows the ex-ante Choquet expected utility levels for these two cases.

Figure 2: Optimal contract vs. no intermediation



The critical degrees of confidence $\gamma_o = \frac{\alpha_2 - \beta_\ell}{\beta_h - \beta_\ell}$ and $\gamma_n = \frac{\alpha_2 - \beta_\ell}{\pi \cdot (\beta_h - \beta_\ell)}$ are obtained where $\beta(\gamma_o \cdot \pi) = \alpha_2 \cdot \pi + (1 - \pi) \cdot \beta_\ell$ and $\beta(\gamma_n \cdot \pi) = \alpha_2$ hold, respectively. The Choquet expected utility in the case of no intermediation forms a lower bound for the ex-ante Choquet expected utility with any kind of voluntary

intermediation. The optimal contract, on the other hand, provides an upper bound for what any intermediary institution can achieve.
In the following sections, we investigate different institutional settings. We compare an asset market for the illiquid asset in Period 1 with a deposit contract offered by a competitive bank.

3 Asset market

As a first institutional environment, suppose that in Period 1 consumers can sell claims to their investment in the illiquid asset. The possibility to trade claims on the asset makes this investment more liquid and provides an extra incentive to invest in it. As before, consumers decide in Period 0 how much of their wealth to invest in the asset and how much to keep as liquid money holdings.

3.1 Market for claims in Period 1

Let us consider first the market for claims to the illiquid asset in Period 1. At this stage, the aggregate money holding M of consumers is given, types are private knowledge and the actual proportion τ of h-types is common knowledge. Since all consumers are identical in Period 0, individual investment policies can be assumed to equal their aggregates.
Denote by q the price of a claim to one unit of the illiquid asset in terms of money. If the price is high enough, consumers of type h, who hold some illiquid asset, will try to sell it in order to benefit from their high value for liquidity β_h. The aggregate supply of such claims is:

$$S(q) = \begin{cases} 1-M & \text{for} \quad q > \frac{\alpha_2}{\beta_\ell} \\ [\tau \cdot (1-M), 1-M] & \text{for} \quad q = \frac{\alpha_2}{\beta_\ell} \\ \tau \cdot (1-M) & \text{for} \quad \frac{\alpha_2}{\beta_\ell} > q > \frac{\alpha_2}{\beta_h} \\ [0, \tau \cdot (1-M)] & \text{for} \quad q = \frac{\alpha_2}{\beta_h} \\ 0 & \text{for} \quad \frac{\alpha_2}{\beta_h} > q \end{cases} .$$

For high prices, $q > \frac{\alpha_2}{\beta_\ell}$, both types of consumers would like to sell their claims. For low prices, $\frac{\alpha_2}{\beta_h} > q$, no one wants to sell them. In the price range $(\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell})$ only h-type consumers want to sell their claims to the illiquid asset. Similarly, consumers of type t want to buy securities for prices below $\frac{\alpha_2}{\beta_t}$. Hence, one obtains the following aggregate demand:

$$D(q) = \begin{cases} 0 & \text{for} \quad q > \frac{\alpha_2}{\beta_\ell} \\ [0, (1-\tau) \cdot \frac{M}{q}] & \text{for} \quad q = \frac{\alpha_2}{\beta_\ell} \\ (1-\tau) \cdot \frac{M}{q} & \text{for} \quad \frac{\alpha_2}{\beta_\ell} > q > \frac{\alpha_2}{\beta_h} \\ [(1-\tau) \cdot \frac{M}{q}, \frac{M}{q}] & \text{for} \quad q = \frac{\alpha_2}{\beta_h} \\ \frac{M}{q} & \text{for} \quad q < \frac{\alpha_2}{\beta_h} \end{cases} .$$

Figure 3: Market for claims to the illiquid asset

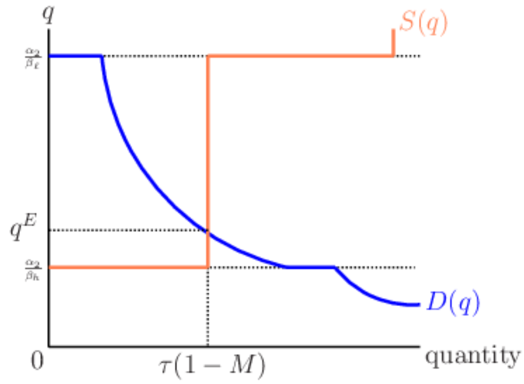


Figure 3 shows these demand and supply curves.
The market for claims clears for a price in the range $[\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell}]$. The equilibrium price q^E depends on the proportion of $h-$types τ and the aggregate investment policy M:

$$q^E(\tau, M) = \begin{cases} \frac{\alpha_2}{\beta_\ell} & \text{if} \quad \tau < \frac{M \cdot \beta_\ell}{M \cdot \beta_\ell + (1-M) \cdot \alpha_2} \\ \frac{1-\tau}{\tau} \cdot \frac{M}{1-M} & \text{if} \quad \tau \in [\frac{M \cdot \beta_h}{M \cdot \beta_h + (1-M) \cdot \alpha_2}, \frac{M \cdot \beta_\ell}{M \cdot \beta_\ell + (1-M) \cdot \alpha_2}] \\ \frac{\alpha_2}{\beta_h} & \text{if} \quad \tau > \frac{M \cdot \beta_h}{M \cdot \beta_h + (1-M) \cdot \alpha_2} \end{cases} . \quad (8)$$

At an equilibrium price $q^E(\tau, M) \in (\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell})$, all consumers of type h sell their claims and all consumers of type ℓ use their money holdings to buy claims.

3.2 Investment decision in Period 0

We now turn to the investment decision in Period 0. Since there is a continuum of consumers, a single consumer's share in the aggregate investment is negligible. Hence, a consumer will take the market price of claims in Period 1, $q^E(\tau, M)$, as given.
Given a price for claims on the illiquid asset of q in Period 1, denote by $R^m(q;t) = \max\{\beta_t, \frac{\alpha_2}{q}\}$ and $R^a(q;t) = \max\{\beta_t \cdot q, \alpha_2\}$ the implicit returns in utility on holding one unit of money or one unit of the illiquid asset, respectively. The indirect utility of a type t consumer who holds m units of money and who expects a price of q for claims to the illiquid asset in Period 1, $\widehat{v}_a(m;q;t)$, is

$$\widehat{v}_a(m;q;t) = m \cdot R^m(q;t) + (1-m) \cdot R^a(q;t).$$

The subscript a of the indirect utility function refers to the institutional framework of an asset market.

A consumer's prediction of the equilibrium asset price $q^E(\tau, M)$ depends on the aggregate money holdings M and the proportion of h-types τ. Hence, indirect utility depends also on these variables. In order to simplify notation, we write $v_a(m; M, \tau, t) = \widehat{v}_a(m; q^E(\tau, M); t)$.
Uncertainty about type and proportion of types is modelled again by the degree of confidence γ which consumers hold in the point expectation π. Hence, one obtains the following Choquet expected indirect utility:

$$\begin{aligned} CEU_a(m; M, \gamma) = \quad & \gamma \cdot [\pi \cdot v_a(m; M, \pi, h) + (1-\pi) \cdot v_a(m; M, \pi, \ell)] \\ & + (1-\gamma) \cdot \min_{(t,\tau) \in \{h,\ell\} \times [0,1]} v_a(m; M, \tau, t). \end{aligned}$$

Consumers choose their initial investment m to maximize $CEU_a(m; M, \gamma)$ given aggregate money holdings M. In an equilibrium, aggregate money holdings M^* must be consistent with individual decisions m^*,

$$M^* = \int_0^1 m^* \ di = m^*.$$

This consistency is equivalent to the assumption that there exists a price q^* for claims in Period 1 which clears the market and produces returns in utility on the two assets $(R^m(q^*; t), R^a(q^*; t))$, which makes consumers indifferent about their initial investment given the uncertainty about (t, τ).
The following theorem shows that an equilibrium exists for any degree of confidence $\gamma \in [0, 1]$.

Proposition 3. Equilibrium in the asset market
There exists a unique equilibrium $(q_a^*(\gamma), M_a^*(\gamma))$ in the asset market satisfying

- $q_a^*(\gamma) = 1$ *and* $M_a^*(\gamma) = \pi,$ *for* $\gamma = 1,$
- $q_a^*(\gamma) \in (\frac{\alpha_2}{\beta_h}, 1)$ *and* $M_a^*(\gamma) \in (\frac{\pi \cdot \alpha_2}{\pi \cdot \alpha_2 + (1-\pi) \cdot \beta_h}, \pi)$ *for* $\gamma \in (\gamma_o, 1),$
- $q_a^*(\gamma) = \frac{\alpha_2}{\beta_h}$ *and* $M_a^*(\gamma) \in [0, \frac{\pi \cdot \alpha_2}{\pi \cdot \alpha_2 + (1-\pi) \cdot \beta_h}].$ *for* $\gamma \in [0, \gamma_o],$

yielding an ex-ante expected utility of

$$V_a^*(\gamma) = \gamma \cdot \pi \cdot q_a^*(\gamma) \cdot \beta_h + (1 - \gamma \cdot \pi) \cdot \alpha_2. \tag{9}$$

Proof. See the appendix. □

In an asset market equilibrium, the price q of claims to the illiquid asset must fulfil a dual role:

- it must clear the market in Period 1 for given holdings of money and the illiquid asset, and
- it must yield equal Choquet expected returns from holding money and from investing in the illiquid asset in Period 0.

If the latter condition were not satisfied, consumers would either hold only money or only the illiquid asset, and no trade would occur in Period 1. We will demonstrate in Section 5 that this dual task impairs the asset market's potential to achieve the optimal allocation in all but the trivial case of an equilibrium without trade. As an institutional arrangement, however, the asset market may dominate the other intermediary institutions.

If there is no ambiguity, for $\gamma = 1$, the asset market price will be $q_a^*(\gamma) = 1$ and the ex-ante expected utility is

$$V_a^*(1) = \pi \cdot \beta_h + (1-\pi) \cdot \alpha_2.$$

As in Diamond (1997)) a secondary market for the illiquid asset can only provide liquidity services but not the optimal cross-subsidy from investors with low liquidity needs to investors with high liquidity needs. Clearly, this provision of liquidity via the secondary claims market improves upon the allocation which a consumer could provide in isolation,

$$V_a^*(1) = \pi \cdot \beta_h + (1-\pi) \cdot \alpha_2 > \max\{\alpha_2\,,\, \gamma \cdot \pi \cdot \beta_h + (1-\gamma \cdot \pi) \cdot \beta_\ell\} = V_n^*(1).$$

but it falls short of the potential payoff possible according to the optimal contract,

$$V_o^*(1) = \frac{\alpha_2 \cdot \pi}{\alpha_2 \cdot \pi + (1-\pi) \cdot \beta_\ell} \cdot \beta_h > \beta_h > \pi \cdot \beta_h + (1-\pi) \cdot \alpha_2 = V_a^*(1).$$

4 Banks

In an alternative institutional arrangement, liquidity is provided by competing banks. Banks collect funds from consumers, invest them jointly and, thus, can provide alternative payouts in the two periods. The instrument to achieve this intertemporal allocation is the deposit contract.

Bank deposit contracts can provide the cross-subsidy required by the optimal contract. In contrast to the secondary asset market, however, deposit contracts are exposed to a risk of coordination failure. If more depositors withdraw their deposits in Period 1 than provided for by the bank, illiquid assets have to be liquidated at the unfavorable rate α_1 in order to fulfill the deposit contract. Excess withdrawals in Period 1 diminish payouts on deposits in Period 2, which may induce long-term depositors to withdraw their funds early. In this section, we will study how ambiguity about aggregate withdrawals affects a consumer's evaluation of the deposit contract.

For a bank deposit contract, the liquidation possibility, $\alpha_1 > 0$, becomes essential. With $\alpha_1 = 0$, the bank would be unable to liquidate long-term investment in favor of early withdrawals. Hence, long-term payoffs would not be affected. There would be no incentive for long-term depositors to withdraw early, even if early withdrawers were to suffer losses on their deposits.[10]

[10]With a secondary market for the illiquid asset, the equilibrium price would determine

4.1 The deposit contract

A deposit contract specifies repayments for both periods according to the following rules:

1. Withdrawals in Period 1 are made on demand. They are treated as senior to withdrawals in Period 2. If withdrawals in Period 1 exceed a bank's reserves, the bank will liquidate part or all of its long-term investment in the illiquid asset in order to satisfy depositors' demand for liquid funds in Period 1.

2. Within each period, consumers have the same priority. If withdrawals exceed the resources of the bank, then consumers calling back their deposits obtain a repayment proportional to their initial deposit.

3. In Period 2 banks distribute their remaining wealth to depositors, who did not withdraw in Period 1.

In Period 0, consumers deposit their wealth with banks. Banks decide on how to invest these funds. Based on their prediction of withdrawals in Period 1, banks hold part of their deposits as reserves in the form of money and invest the remainder in the illiquid asset. This policy guarantees the contracted repayments in both periods, provided the bank predicts withdrawals correctly and does not have to resort to the liquidation of illiquid assets. Free entry and competition among banks about the terms of deposit contracts ensures zero profits. It also guarantees an investment policy in the interest of depositors[11]. These assumptions allow us to portray the competing banks by a representative bank.

Formally, the deposit contract of a bank is characterized by the interest rates (i_1, i_2) promised for Periods 1 and 2, respectively. Since liquidation of funds invested in the illiquid asset is costly, the bank holds reserves R equal to its payments predicted for Period 1. If the fraction W_0 of depositors withdraws their funds in Period 1, the bank has to pay out $(1+i_1)\cdot W_0$. Hence, the bank must hold reserves in terms of money equal to $R = (1+i_1)\cdot W_0$. Remaining deposits, $1-R$, will be invested in the illiquid asset. In Period 2, competition forces the bank to pay out all its returns from investment, $\alpha_2 \cdot (1-R)$, to depositors who did not withdraw in Period 1. With an initial amount of deposits equal to 1, the zero-profit condition, $(1+i_2)\cdot(1-W_0) = \alpha_2\cdot(1-R)$, determines the interest rate i_2. Hence, interest rates (i_1, i_2) are functions of the bank's reserve policy R and predicted withdrawals W_0,

$$i_1(R, W_0) = \frac{R}{W_0} - 1, \qquad i_2(R, W_0) = \alpha_2 \cdot \frac{1-R}{1-W_0} - 1. \tag{10}$$

the liquidation rate endogenously. Studying a secondary market for the illiquid asset in the presence of the bank deposit contract, as in Diamond (1997), would exceed the scope of one paper.

[11] Competition among banks is in the spirit of Allen and Gale (1998). For a more extensive discussion in a similar context we refer to Spanjers (1999, Chapter 3).

It is, however, the *actual fraction of withdrawals* in Period 1, W, together with the priority rules specified above, which determine the actual payoffs of deposits, $\rho_1(W; R, W_0)$ and $\rho_2(W; R, W_0)$, as

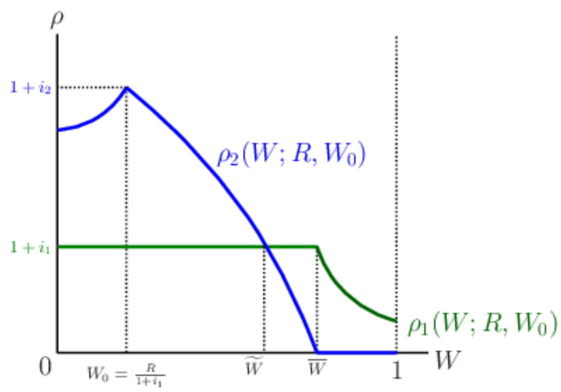
$$\rho_1(W; R, W_0) = \min\{\frac{R}{W_0}, \frac{R + \alpha_1 \cdot (1 - R)}{W}\} \tag{11}$$

and

$$\begin{aligned} &\rho_2(W; R, W_0) \\ &= \max\left\{0, \frac{\alpha_2}{1 - W} \cdot \left[(1 - R) + \frac{1}{\alpha_2} \max\{W_0 - W, 0\} - \frac{1}{\alpha_1} \cdot \frac{R}{W_0} \cdot \max\{W - W_0, 0\}\right]\right\}. \end{aligned} \tag{12}$$

Equation (11) reflects the priority rule that returns on deposits in Period 1 will be maintained as long as possible, i.e., as long as $[1 + i_1(R, W_0)] \cdot W = \frac{R}{W_0} \cdot W$ is less than $R + \alpha_1 \cdot (1 - R)$, the maximal amount of liquidity a bank can raise in Period 1. The actual return in Equation (12) follows from the assumption that banks will distribute all their funds in Period 2.
Figure 4 shows the actual returns of deposits as a function of the withdrawals in Period 1.

Figure 4: Actual returns of deposits



For $W = W_0$, actual returns equal the promised returns, $\rho_1(W_0; R, W_0) = 1 + i_1(R, W_0)$ and $\rho_2(W_0; R, W_0) = 1 + i_2(R, W_0)$. Moreover, let $\overline{W}$ be the aggregate level of withdrawals for which all assets have to be liquidated in order to maintain the return on deposits in Period 1, $\overline{W} = \frac{R + \alpha_1 \cdot (1 - R)}{R} \cdot W_0$, then $\rho_2(\overline{W}; R, W_0) = 0$ holds.
According to the assumptions on asset returns made in Assumption 1, banks will not offer interest rates on a deposit contract such that neither consumers with high liquidity needs nor consumers with low liquidity demand will withdraw their deposits in Period 1. Hence, banks will choose a reserve policy R, and associated interest rates, such that only consumers with high liquidity demand will withdraw their deposits in Period 1. With such a

deposit contract banks must prepare for withdrawals equal to the proportion of consumers with high liquidity needs τ.
We assume that banks have *rational expectations* regarding the proportion of consumers with high liquidity needs. By Assumption 2, this implies

$$W_0 = \int \tau \; dp(\tau) = \pi.$$

Given believes about aggregate withdrawals in Period 1, $W_0 = \pi$, the bank's reserve policy R, and the implied deposit interest rates $(i_1(R,\pi), i_2(R,\pi))$ according to Equation (10), must guarantee that only consumers with high liquidity needs will want to withdraw their funds in Period 1,

$$\beta_h \cdot (1 + i_1(R,\pi)) \geq (1 + i_2(R,\pi)) \geq \beta_\ell \cdot (1 + i_1(R,\pi)) . \qquad (13)$$

Equation (13) shows the incentive compatibility constraints which a bank's reserve policy R must satisfy.

4.2 The depositor's problem

Consider a consumer who has deposited all funds with the bank in Period 0. In Period 1 consumers learn their types, and the aggregate demand for liquidity τ becomes known as well. Type-h consumers will withdraw their funds if $\rho_2(W;R,\pi) \leq \beta_h \cdot \rho_1(W;R,\pi)$ holds. Otherwise they will leave their deposits in the bank. Similarly, consumers of type ℓ will *not* withdraw their deposits for $\beta_\ell \cdot \rho_1(W;R,\pi) \leq \rho_2(W;R,\pi)$. Hence, one can summarize the aggregate withdrawal behavior by

$$\mathcal{W}(W;R) = \begin{cases} 1 & \text{if } \rho_2(W;R,\pi) < \beta_\ell \cdot \rho_1(W;R,\pi) \\ [\tau,1] & \text{if } \rho_2(W;R,\pi) = \beta_\ell \cdot \rho_1(W;R,\pi) \\ \tau & \text{if } \beta_\ell \cdot \rho_1(W;R,\pi) < \rho_2(W;R,\pi) < \beta_h \cdot \rho_1(W;R,\pi) \\ [0,\tau] & \text{if } \beta_h \cdot \rho_1(W;R,\pi) = \rho_2(W;R,\pi) \\ 0 & \text{if } \beta_h \cdot \rho_1(W;R,\pi) < \rho_2(W;R,\pi) \end{cases} .$$

Aggregate withdrawals W^* are a Nash equilibrium if they are a fixed point of $\mathcal{W}(W^*;R)$, i.e. if $\mathcal{W}(W^*;R) = W^*$.
Figure 4 shows the return functions $\rho_1(W;R,\pi)$ and $\rho_2(W;R,\pi)$. For the special case of $\beta_\ell = 1$, one can use this diagram to check for which levels of W there is an equilibrium. Given our assumptions on the asset payouts, the return functions intersect just once at the level of withdrawals $\widetilde{W}$.
In general, the critical level of withdrawals $\widetilde{W}$ occurs if the proportion of consumers with high liquidity needs exceeds the bank's reserves such that second-period returns on deposits ρ_2 fall to a level where consumers with low liquidity needs become indifferent between withdrawing and leaving their deposits in the bank, $\beta_\ell \cdot \rho_1(\widetilde{W};R,\pi) = \rho_2(\widetilde{W};R,\pi)$.

Hence, it is obvious that there are two types of equilibria[12]:

$$\text{(i)} \qquad \mathcal{W}(\tau; R) = \tau, \qquad \textit{regular equilibrium}$$

$$\text{(ii)} \qquad \mathcal{W}(1; R) = 1. \qquad \textit{bank-run equilibrium}$$

Given a bank's reserve policy R, it is clear from Figure 5 that $W^* = \tau$ is a Nash equilibrium if $\tau \leq \widetilde{W}$ holds, otherwise $W^* = 1$ is the unique equilibrium.

Figure 5: Withdrawal equilibria

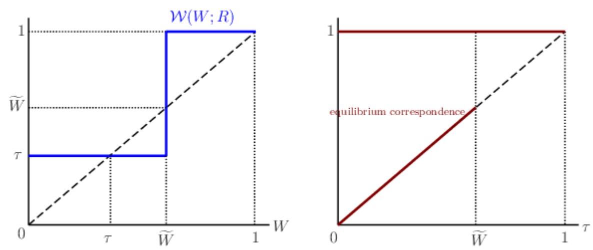


Figure 5 shows in its left part a typical equilibrium constellation. The critical value $\widetilde{W}$ is a function of the bank's reserve policy R. In the right part of Figure 5, the equilibrium correspondence for varying proportions τ of consumers with high liquidity needs is displayed. We will demonstrate in the next section, that the bank will always choose a reserve policy which guarantees that the rationally expected withdrawals $W_0 = \pi$ are less than $\widetilde{W}$.

In a regular equilibrium, only consumers with high liquidity needs will withdraw their deposits in Period 1, while in the bank-run equilibrium all depositors will withdraw their funds. The bank-run equilibrium always exists. In contrast, a regular equilibrium exists only if aggregate liquidity needs τ are not too high. In line with the literature[13], we will assume that the regular equilibrium $W^* = \tau$ obtains whenever it exists.

By choosing to withdraw their deposits or to leave them with the bank in Period 1, consumers of type t can obtain the utility

$$v_b(W; R, t) = \max\{\beta_t \cdot \rho_1(W; R, \tau), \rho_2(W; R, \tau)\}. \tag{14}$$

In Period 0, consumers face uncertainty about their type and the aggregate demand for liquidity τ. The Choquet expected utility of a deposit contract in Period 0 is given in the following lemma.

[12] Strictly speaking, $\mathcal{W}(W; R, W_0)$ is a correspondence and there is also a mixed strategy equilibrium $\widetilde{W}$, where *all* consumers with high liquidity needs and *some* consumers with low liquidity needs withdraw their deposits. This mixed equilibrium, which we disregard, is obtained for $\beta_\ell \cdot \rho_1(\widetilde{W}; R, W_0) = \rho_2(\widetilde{W}; R, W_0)$.

[13] See, for example, Diamond (1997) or Allen and Gale (2004).

Lemma 1. *For a bank choosing reserve policy R in Period 0, the Choquet expected utility of a consumer from a deposit contract is*

$$CEU_b(R;\gamma) = [\gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell] \cdot R + [\gamma \cdot \alpha_2 + (1-\gamma) \cdot \beta_\ell \cdot \alpha_1] \cdot (1-R). \tag{15}$$

Proof. See the appendix. □

4.3 Banks' reserve policy

When choosing their reserve policy R banks implicitly also determine the interest rates on deposits (Equation (10)). Competition forces banks to make this choice in the interest of consumers. Hence, banks will choose R such that the consumers' ex-ante Choquet expected utility $CEU_b(R;\gamma)$, derived in Lemma 1, is maximized, subject to the constraint that consumers with low liquidity needs do not withdraw funds in Period 1, Equation (13). As solution of the decision problem,

$$\begin{aligned} &\text{choose } R \text{ to maximize} && CEU_b(R;\gamma) \\ &\text{subject to} && 1 + i_2(R,\pi) \geq \beta_\ell \cdot [1 + i_1(R,\pi)], \end{aligned}$$

one obtains the optimal reserve policy R^*. From Equation (15) and Assumption 1, it is immediately clear that $CEU_b(R;\gamma)$ is a strictly increasing function of R. Since $i_2(R,\pi)$ is strictly decreasing and $i_1(R,\pi)$ is strictly increasing in R, the constraint $1 + i_2(R^*,\pi) = \beta_\ell \cdot [1 + i_1(R^*,\pi)]$ must be binding. Substituting from Equation (10), one obtains the following Lemma.

Lemma 2. *If all consumers (voluntarily) deposit their wealth with the bank, then the optimal reserve holdings are*

$$R^* = \frac{\alpha_2 \cdot \pi}{\alpha_2 \cdot \pi + (1-\pi) \cdot \beta_\ell}. \tag{16}$$

Notice that the optimal reserve holdings R^* do not depend on the degree of confidence γ. Moreover, the optimal reserve holdings R^* equal the aggregate money holdings of the optimal contract $M_o^*(\gamma)$, derived in Section 2.2.2. Substituting the optimal reserve policy R^* from Equation (16) into the Choquet expected utility function $CEU(R,\gamma)$, Equation (15), yields

$$CEU_b(R^*;\gamma) = \frac{\pi \cdot \alpha_2 \cdot [\gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell] + (1-\pi) \cdot \beta_\ell \cdot [\gamma \cdot \alpha_2 + (1-\gamma) \cdot \beta_\ell \cdot \alpha_1]}{\alpha_2 \cdot \pi + (1-\pi) \cdot \beta_\ell} \tag{17}$$

Choosing its reserves appropriately, the bank can design a deposit contract which guarantees consumers not just liquidity but also the cross subsidy required by the optimal contract. In fact, if there is no uncertainty, $\gamma = 1$, then a bank deposit contract will achieve the same ex-ante expected utility as the optimal contract,

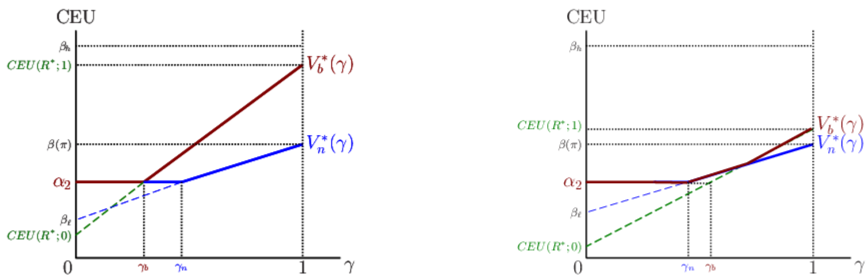
$$CEU_b(R^*;1) = \frac{\alpha_2 \cdot [\pi \cdot \beta_h + (1-\pi) \cdot \beta_\ell]}{\alpha_2 \cdot \pi + (1-\pi) \cdot \beta_\ell} = V_o^*(1).$$

In Period 0, consumers can choose between a deposit contract and direct investment in the assets as analyzed in Section 2.1. They will deposit their funds in a bank if the Choquet expected utility of the deposit contract $CEU_b(R^*;\gamma)$ exceeds the Choquet expected return from direct investment, $V_n^*(\gamma)$. A bank deposit contract allows consumers therefore to obtain an ex-ante Choquet expected utility

$$V_b^*(\gamma) = \max\{V_n^*(\gamma), CEU_b(R^*;\gamma)\}.$$

Figure 6 shows $V_n^*(\gamma) = \max\{\alpha_2\,,\, \gamma\cdot\pi\cdot\beta_h + (1-\gamma\cdot\pi)\cdot\beta_\ell\}$ and $CEU_b(R^*;\gamma)$.

Figure 6: CEU of deposit contact



Let γ_b denote the level of confidence for which $V_n^*(\gamma)$ equals $CEU_b(R^*;\gamma)$. In the left diagram, the case $CEU_b(R^*;\gamma_b) = \alpha_2 > \gamma_b\cdot\pi\cdot\beta_h + (1-\gamma_b\cdot\pi)\cdot\beta_\ell$ is illustrated. In this case, we have $\gamma_b < \gamma_n$. The right diagram of Figure 6 shows the case $CEU_b(R^*;\gamma_b) = \alpha_2 < \gamma_b\cdot\pi\cdot\beta_h + (1-\gamma_b\cdot\pi)\cdot\beta_\ell$ and $\gamma_b > \gamma_n$. Lemma 3 provides a formal proof.

Lemma 3. *There is a unique value $\gamma_b \in (0,1)$ for which $V_n^*(\gamma_b) = CEU_b(R^*;\gamma_b)$ holds. Moreover, $\gamma_b > \gamma_o$.*

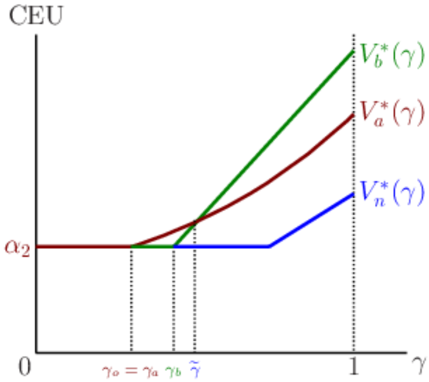
Proof. See the appendix. □

5 Banks or asset markets?

It remains to compare the ex-ante Choquet expected utility of the allocations achieved with a secondary asset market and with a bank deposit contract. Depending on the degree of confidence consumers' preferences for these institutions of financial intermediation vary.
Recall that $\gamma_o = \frac{\alpha_2-\beta_\ell}{\beta_h-\beta_\ell}$ is the degree of confidence below which the optimal contract recommends to invest all funds in the illiquid asset. The following result provides a full comparison of these two institutions[14].

[14] In Spanjers (1999), *mutual funds* are also considered. If asset-backed short-selling is not permitted, a single mutual fund can implement the optimal outcome. In the more plausible case, however, that short-selling is allowed or the financial system consists of a number of competing mutual funds, the outcome of the asset market is obtained.

Figure 7: Evaluation of intermediary institutions



Theorem 1. *There exists a level of confidence $\widetilde{\gamma} \in (\gamma_o, 1)$ such that,*

(i)	*for $\gamma \leq \gamma_o$,*	*consumers need* no intermediation *at all,*
(ii)	*for $\gamma_o < \gamma < \widetilde{\gamma}$,*	*consumers strictly prefer the* secondary asset market*,*
(iii)	*for $\gamma > \widetilde{\gamma}$,*	*consumers strictly prefer the* bank deposit contract*.*

Proof. See the appendix. □

Figure 7 illustrates this result.

Roughly speaking, three cases can arise. For degrees of confidence below γ_o, neither bank deposit contracts nor a secondary asset market can improve upon the investment opportunities without intermediation. For a range of intermediate degrees of confidence, the secondary asset market dominates the bank deposit contract offered by competing banks, while for high degrees of confidence, consumers prefer the outcome of bank deposit contracts over the secondary asset market allocation.

To obtain more insight into the reason for this ranking, consider the case of no ambiguity, $\gamma = 1$. The second column of the table below gives the *ex-ante expected utility* obtained with these institutions. The fourth column indicates the state in which the worst utility occurs.

	Expected utility	*Worst utility*	*Worst state*
No intermediation	$\pi \cdot \beta_h + (1-\pi) \cdot \beta_\ell$	α_2	$(t, \tau) = (\ell, 0)$
Asset market	$\pi \cdot \beta_h + (1-\pi) \cdot \alpha_2$	α_2	$(t, \tau) = (\ell, 0)$
Bank deposits	$R^* \cdot \beta_h + (1-R^*) \cdot \alpha_2$	$\beta_\ell \cdot [R^* + (1-R^*) \cdot \alpha_1]$	$(t, \tau) = (\ell, 1)$
Optimal contract	$M_o^* \cdot \beta_h + (1-M_o^*) \cdot \alpha_2$	α_2	$(t, \tau) = (\ell, 0)$

With no ambiguity and full confidence, the secondary asset market guarantees consumers full liquidity. Despite their ignorance about their individual liquidity needs, consumers obtain the same ex-ante expected utility as if they invested their funds in the illiquid asset and withdrew at par if they would turn out to be high types β_h. The worst case of a secondary asset market occurs if all consumers have low liquidity needs, $\tau = 0$. Notice that

the result of the asset market does not depend on any institutional investment activity. This is the reason why the asset market cannot engineer the optimal cross subsidy from ℓ-types to h-types.
With bank deposit contracts, the banks' investment policy R^* becomes crucial. Notice that the optimal reserve holdings of a bank do not depend on the degree of confidence. Moreover, the optimal investment policy of a bank must mimic the investment of the optimal contract:

$$R^* = M_o^* = \frac{\pi \cdot \alpha_2}{\pi \cdot \alpha_2 + (1-\pi) \cdot \beta_\ell}.$$

While this investment policy can achieve the same ex-ante Choquet expected utility as the optimal contract, if consumers have full confidence in the probability distributions over individual and aggregate liquidity needs, it makes the deposit contract delicately poised on the accurate prediction of aggregate liquidity needs π. In particular, for the deposit contract, the worst case is the state where all consumers have high liquidity needs, $\tau = 1$, since banks would be forced to liquidate their illiquid investment prematurely. Hence, the worst case of a deposit contract $\beta_\ell \cdot [R^* + (1 - R^*) \cdot \alpha_1] < \beta_\ell$ is worse than simply holding money.
As the degree of confidence runs from 1 to 0 the utility of the worst case becomes more important relative to the expected utility. Hence, we find that consumers' evaluation of financial institutions changes. There is a switch-over point $\widetilde{\gamma}$ below which the expected utility ranking of banks and the asset market is reversed.

6 Concluding remarks

We have looked at our results from the point of view of a consumer assessing different institutions of financial intermediation. We have seen that this assessment depends crucially on the consumers' degree of ambiguity with respect to individual and aggregate liquidity needs.
There are more general lessons of our analysis. The aggregate results of any institution of financial intermediation must be judged not only according to its expected outcomes but also according to its worst outcome. These rankings may vary greatly. What makes one institution ideal, if uncertainty is low, may make it vulnerable if uncertainty is high. Bank deposit contracts require an active investment policy of banks. This opens up opportunities for a better outcome than pure liquidity provision. An inappropriate reserve policy, on the other hand, may do great harm.
Ambiguity about what "adequate reserve holdings" are, together with the spread between normal expected outcomes and the outcome under a worst-case scenario, may also guide regulators of financial institutions. Moreover, since financial intermediaries may take part in several institutions, their

choice is likely to be influenced by their ambiguity regarding critical parameters such as aggregate liquidity needs.

Of course, there are several important aspects neglected in our analysis. Probably the most important shortcoming is the assumption of certain payoffs for the illiquid asset. This assumption is responsible for the preference of consumers for the long-term asset if ambiguity is high. With some uncertainty about the long-term asset's returns this preference would disappear. Moreover, a distinction between illiquidity issues and insolvency issues appears possible. Given the novelty of the uncertainty model in the context of financial intermediation models, however, we wanted to stay as close as possible to the existing liquidity models in the tradition of Diamond and Dybvig (1983) which is well understood for the case of pure risk.

Several other questions remain beyond the scope of this paper. A careful analysis as to whether bank deposit contracts can coexist with asset markets if agents have access to both institutions. Since we confirm the results of Diamond (1997)) for the case of expected utility, it is likely that one may have to consider constraints on market participation even with ambiguity. Ambiguity may, however, also provide a reason why consumers may choose alternative financial intermediary institutions. In an economy where the degree of confidence differs across the population, ranging from consumers facing extremely low degrees of ambiguity to those with very high degrees of ambiguity, one may expect to see bank deposit contracts coexist with asset markets. Consumers with low degrees of confidence prefer the asset market, while consumers with high degrees of confidence may favour the bank deposit contract.

In still another institutional setting, one may find consumers with high degrees of confidence providing equity for banks that offer deposit contracts, effectively insuring the consumers with low degrees of confidence. Such result would be in line with the findings of Diamond (1997) for risk-averse consumers in the absence of ambiguity.

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Appendices

A. Ambiguous beliefs

Ambiguity in beliefs is modelled by a special case of a Choquet expected utility approach (Schmeidler, 1989) which allows also for an interpretation in terms of multiple priors (Gilboa and Schmeidler, 1989). Beliefs are modelled by simple capacities, a special case of neo-additive capacities, which were axiomatised and discussed in detail in Chateauneuf *et al.* (2007). Eichberger and Kelsey (2009) provides a survey of different models of ambiguity and ambiguity attitudes. Eichberger *et al.* (2009) apply neo-additive capacities to other models. In this appendix, we will provide a short, intuitive introduction to CEU with simple capacities.

Consider a state space S which is a compact and convex subset of $\mathbb{R}^n$ and let p be an additive probability distribution on S. A simple capacity $\nu(\gamma, p)$, parametrised by the additive probability distribution p and the confidence parameter γ is the set function defined as

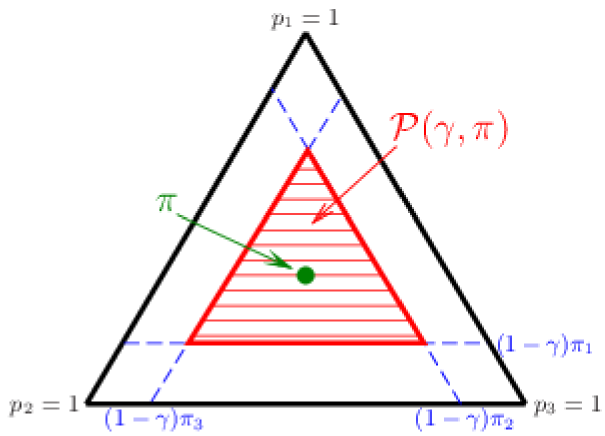
$$\nu(E; \gamma, p) = \begin{cases} \gamma \cdot p(E) & \text{for} \quad E \subset S \\ \\ 1 & \text{for} \quad E = S \end{cases}$$

for any p-measurable set E and some $\gamma \in [0, 1]$.

Simple capacities provide an intuitive notion of ambiguity. The capacity value $\nu(E)$ can be interpreted as the lowest probability for an event E which a decision maker considers possible.

For the case of only three states $S = \{s_1, s_2, s_3\}$, one can represent the set of probabilities by the set of probability distribuitons $P(\gamma, \pi)$ in Figure 8.

Figure 8: Multiple priors of a simple capacity



The solid point in the triangel represents the additive probability distribution $\pi = (\pi_1, \pi_2, \pi_3)$ on which the capacity is focused. For $\gamma = 1$ the simple capacity coincides with the single additive probability distribution π, i.e. $P(1,\pi) = \{\pi\}$. At the other extreme, for $\gamma = 0$, the decision maker considers all probabilities as possible and the set of probability distributions $P(0,\pi)$ coincides with the full simplex. Intermediate cases, $\gamma \in (0,1)$, correspond to triangular sets such as $P(\gamma,\pi)$ represented in red in Figure 8. Hence, one can interpret the parameter γ as the degree of confidence in the additive probability distribution p. Ambiguity increases as confidence falls, $\gamma \to 0$.

Simple capacities maintain many properties of additive probability distributions and have a natural interpretation in terms of beliefs. Ambiguity is simply the counterpart of the decision maker's confidence in a probabilistic assessment. Moreover, simple capacities allow for a simple and intuitive decision criterion for state-contingent outcome functions, i.e., measurable functions $f : S \to \mathbb{R}$.

The Choquet integral of f with respect to a simple capacity $\nu(\gamma,p)$, or Choquet expected utility of f, $CEU(f)$, is the convex combination with weight γ of the expected utility of f with respect to the probability distribution p and the worst outcome of f on S. The following result is proved in Eichberger and Kelsey (1999, Proposition 2.1).

Proposition 4. *Choquet integral of a simple capacity*
Consider a simple capacity $\nu(\gamma,p)$ with degree of confidence γ and additive probability distribution p. The Choquet integral $CEU(f,\gamma,p)$ of a continuous function f with repect to the simple capacity has the following form:

$$CEU(f,\gamma,p) = \gamma \cdot \int_S f \; dp + (1-\gamma) \cdot \min_{s \in S} f(s). \tag{18}$$

Proposition 4 offers an intuitive and parsimonious preference representation[15] of a decision maker facing ambiguity about p. For the case of full confidence, $\gamma = 1$, one has the familiar expected utility form. As ambiguity increases and confidence in p falls, $\gamma \to 0$, more weight is given to the worst outcome of f on S. For $\gamma = 0$, the maximin decision rule obtains. Notice that the Choquet integral represents a pessimistic attitude towards ambiguity. The more ambiguity a decision maker experiences, the more weight will be put on the worst of the state-contingent outcomes of an action[16].

B. Proofs

[15] A simple capacity is a special case of a NEO-additive capacity which has been axiomatised in Chateauneuf, Eichberger and Grant (2007).

[16] Eichberger and Kelsey (2014) provide a survey over alternative ways of modelling ambiguity.

Proof of Proposition 3. *Equilibrium in the asset market*
There exists a unique equilibrium $(q_a^*(\gamma), M_a^*(\gamma))$ in the asset market satisfying

$$\begin{array}{lllll} \text{(i)} & q_a^*(\gamma) = 1 & \text{and} & M_a^*(\gamma) = \pi, & \text{for} \quad \gamma = 1, \\ \text{(ii)} & q_a^*(\gamma) \in (\frac{\alpha_2}{\beta_h}, 1) & \text{and} & M_a^*(\gamma) \in (\frac{\pi \cdot \alpha_2}{\pi \cdot \alpha_2 + (1-\pi) \cdot \beta_h}, \pi) & \text{for} \quad \gamma \in (\gamma_o, 1), \\ \text{(iii)} & q_a^*(\gamma) = \frac{\alpha_2}{\beta_h} & \text{and} & M_a^*(\gamma) \in [0, \frac{\pi \cdot \alpha_2}{\pi \cdot \alpha_2 + (1-\pi) \cdot \beta_h}]. & \text{for} \quad \gamma \in [0, \gamma_o], \end{array}$$

yielding an ex-ante expected utility of

$$V_a^*(\gamma) = \gamma \cdot \pi \cdot q_a^*(\gamma) \cdot \beta_h + (1 - \gamma \cdot \pi) \cdot \alpha_2.$$

Proof. The proof is organized as follows: In Part A, we transform the problem in a more suitable equivalent form. In Part B, we consider two parameter constellations and derive the equilibria.
Part A: Transformation of the problem
Let

$$\begin{aligned} \hat{v}_a(m; q, t) &= m \cdot R^m(q; t) + (1 - m) \cdot R^a(q; t) \\ &= m \cdot \max\{\beta_t, \frac{\alpha_2}{q}\} + (1 - m) \cdot \max\{\alpha_2, \beta_t \cdot q\}. \end{aligned} \tag{19}$$

For all $q \in [\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell}]$ we have $\hat{v}_a(m; q, h) > \hat{v}_a(m; q, \ell)$. Furthermore, $\hat{v}_a(m; q, \ell)$ is decreasing in q. From $q^E(\tau, M) \leq \frac{\alpha_2}{\beta_\ell}$ it now follows that

$$\begin{aligned} &\min_{(t,\tau) \in \{h,\ell\} \times [0,1]} \hat{v}_a(m; q_a^E(\tau, M); t) \\ &= \min_{q \in [\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell}]} \hat{v}_a(m; q; \ell) = m \cdot \beta_\ell + (1 - m) \cdot \alpha_2. \end{aligned} \tag{20}$$

Define the function $B(m, q)$ as

$$B(m, q) = \gamma \cdot [\pi \cdot \hat{v}_a(m; q, h) + (1 - \pi) \cdot \hat{v}_a(m; q, \ell)] + (1 - \gamma) \cdot \min_{q \in [\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell}]} \hat{v}_a(m; q; \ell). \tag{21}$$

Using Equation (19) and Equation (20) it is easy to check that, for $q \in$

$[\frac{\alpha_2}{\beta h}, \frac{\alpha_2}{\beta_\ell}]$, the following equalities hold:

$$\begin{aligned}
B(m, q; \gamma) &= \gamma \cdot \left\{ \pi \cdot \left[m \cdot \max\{\beta_h, \frac{\alpha_2}{q}\} + (1-m) \cdot \max\{\alpha_2, \beta_h \cdot q\} \right] \right. \\
&\left. +(1-\pi) \cdot \left[m \cdot \max\{\beta_\ell, \frac{\alpha_2}{q}\} + (1-m) \cdot \max\{\alpha_2, \beta_\ell \cdot q\} \right] \right\} \\
&+ (1-\gamma) \cdot [m \cdot \beta_\ell + (1-m) \cdot \alpha_2] \\
&= \gamma \cdot \left\{ \pi \cdot [m \cdot \beta_h + (1-m) \cdot q \cdot \beta_h\}] + (1-\pi) \cdot \left[m \cdot \frac{\alpha_2}{q} + (1-m) \cdot \alpha_2\} \right] \right\} \\
&+ (1-\gamma) \cdot [m \cdot \beta_\ell + (1-m) \cdot \alpha_2] \\
&= m \cdot \left[\gamma \cdot \left(\pi \cdot \beta_h + (1-\pi) \cdot \frac{\alpha_2}{q} \right) + (1-\gamma) \cdot \beta_\ell \right] \\
&+ (1-m) \cdot [\gamma \cdot (\pi \cdot q \cdot \beta_h + (1-\pi) \cdot \alpha_2) + (1-\gamma) \cdot \alpha_2] \\
&= m \cdot \hat{v}^m(q; \gamma) + (1-m) \cdot \hat{v}^a(q; \gamma)
\end{aligned}$$

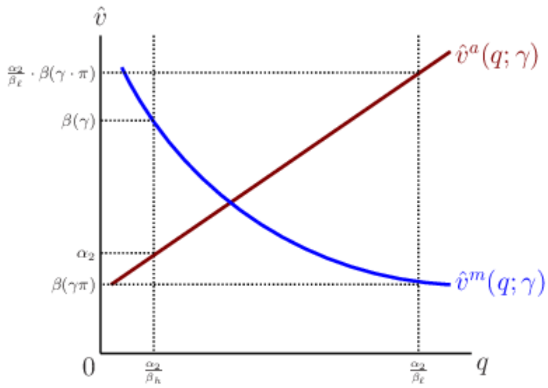
with

$$\hat{v}^m(q; \gamma) = \left[\gamma \cdot \left(\pi \cdot \beta_h + (1-\pi) \cdot \frac{\alpha_2}{q} \right) + (1-\gamma) \cdot \beta_\ell \right], \tag{22}$$

$$\hat{v}^a(q; \gamma) = [\gamma \cdot (\pi \cdot q \cdot \beta_h + (1-\pi) \cdot \alpha_2) + (1-\gamma) \cdot \alpha_2]. \tag{23}$$

Figure9 shows these functions.

Figure 9: $\hat{v}^m(q; \gamma)$ and $\hat{v}^a(q; \gamma)$



Recalling $v_a(m; M, \tau, t) = \widehat{v}_a(m; q^E(\tau, M); t)$, one checks easily that the Choquet expected utility satisfies:

$$CEU_a(m; M, \gamma) = B(m, q^E(\pi, M); \gamma).$$

Hence, (M^*, q^*) is an equilibrium, if M^* maximizes $B(M, q^E(\pi, M); \gamma)$ or, equivalently, if the best-response correspondence $\arg\max_M B(M, q^E(\pi, M); \gamma)$

has a fixed point:

$$M^* \in \arg\max_M B(M^*, q^E(\pi, M^*); \gamma)$$
$$= \begin{cases} \{0\} & \text{for} \quad \hat{v}^m(q^E(\pi, M^*); \gamma) < \hat{v}^a(q^E(\pi, M^*); \gamma) \\ [0,1] & \text{for} \quad \hat{v}^m(q^E(\pi, M^*); \gamma) = \hat{v}^a(q^E(\pi, M^*); \gamma) \\ \{1\} & \text{for} \quad \hat{v}^m(q^E(\pi, M^*); \gamma) > \hat{v}^a(q^E(\pi, M^*); \gamma) \end{cases} .$$

Part B: Equilibria
Case (i): Assume that $\beta(\gamma) = \gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell > \alpha_2$ holds.
In this case neither $M^* = 0$ nor $M^* = 1$ can be equilibria. *(a)* Suppose $M^* = 0$. Then, from the analysis of the market for claims in Section 3.1, there must be q such $S(q) = 0$. Hence, $q \leq \frac{\alpha_2}{\beta_h}$ must be true. From Equations (22) and (23) it follows that for $q \leq \frac{\alpha_2}{\beta_h}$

$$\hat{v}^m(q;\gamma) \geq \hat{v}^m(\frac{\alpha_2}{\beta_h};\gamma) = \beta(\gamma) > \alpha_2 = \hat{v}^a(\frac{\alpha_2}{\beta_h};\gamma) \geq \hat{v}^a(q;\gamma)$$

holds, if $\beta(\gamma) = \gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell > \alpha_2$. Hence, for $\beta(\gamma) > \alpha_2$, one has $M^* = 1$, a contradiction.
(b) Suppose $M^* = 1$. Then, from demand and supply for claims to the illiquid asset in Section 3.1, there must be q such $D(q) = 0$. Hence, $q \geq \frac{\alpha_2}{\beta_\ell}$ must hold. □

From Equations (22) and (23) it follows for $q \geq \frac{\alpha_2}{\beta_\ell}$ that

$$\hat{v}^m(q;\gamma) \leq \hat{v}^m(\frac{\alpha_2}{\beta_\ell};\gamma) = \beta(\gamma \cdot \pi) < \frac{\alpha_2}{\beta_\ell} \cdot \beta(\gamma \cdot \pi) = \hat{v}^a(\frac{\alpha_2}{\beta_\ell};\gamma) \leq \hat{v}^a(q;\gamma).$$

Thus, $M^* = 0$, a contradiction.
(c) From (a) and (b) it follows that $M^* \in (0,1)$. Therefore $\hat{v}^m(q;\gamma) - \hat{v}^a(q;\gamma) = 0$ must hold. This equation is equivalent to the quadratic equation

$$[\gamma \cdot \pi \cdot \beta_h] \cdot q^2 + [\alpha_2 - \gamma \cdot \pi \cdot (\alpha_2 + \beta_h) - (1-\gamma) \cdot \beta_\ell)] \cdot q - [\gamma \cdot (1-\pi) \cdot \alpha_2] = 0. \tag{24}$$

Since $\gamma \cdot (1-\pi) \cdot \alpha_2 > 0$ holds, Equation (24) has a unique positive solution $q_a^*(\gamma) \in (\frac{\alpha_2}{\beta_h}, 1]$. It requires a straightforward calculation to check that $q_a^*(1) = 1$ and $q_a^*(\gamma_o) = \frac{\alpha_2}{\beta_h}$, where $\gamma_o = \frac{\alpha_2 - \beta_\ell}{\beta_h - \beta_\ell}$. Applying the implicit function theorem to $\hat{v}^m(q;\gamma) - \hat{v}^a(q;\gamma) = 0$, one checks easily that $q_a^*(\gamma)$ is strictly increasing in γ.
At the price $q_a^*(\gamma)$ consumers are indifferent about their individual money holdings m. From Equation (8) in Section 3.1 one has $q_a^*(\gamma) = \frac{1-\pi}{\pi} \cdot \frac{M_a^*(\gamma)}{1-M_a^*(\gamma)}$ and therefore obtains

$$M_a^*(\gamma) = \frac{\pi \cdot q_a^*(\gamma)}{\pi \cdot q_a^*(\gamma) + (1-\pi)}$$

as the aggregate money holdings.
Moreover, from Equations (22) and (23), one gets

$$\begin{aligned} V_a^*(\gamma) &= CEU_a(M_a^*(\gamma), M_a^*(\gamma); \gamma) \\ &= \hat{v}^m(q_a^*(\gamma); \gamma) = \hat{v}^a(q_a^*(\gamma); \gamma) \\ &= \gamma \cdot [\pi \cdot q_a^*(\gamma) \cdot \beta_h + (1-\pi) \cdot \alpha_2] + (1-\gamma) \cdot \alpha_2 \\ &= \gamma \cdot \pi \cdot q_a^*(\gamma) \cdot \beta_h + (1 - \gamma \cdot \pi) \cdot \alpha_2. \end{aligned}$$

This proves (i) and (ii) of the proposition.
Case (ii): Assume now that $\beta(\gamma) = \gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell \leq \alpha_2$ holds.
From Equations (22) and (23) it follows that

$$\hat{v}^m(\frac{\alpha_2}{\beta_h}; \gamma) = \beta(\gamma) \leq \alpha_2 = \hat{v}^a(\frac{\alpha_2}{\beta_h}; \gamma).$$

For $\beta(\gamma) < \alpha_2$, one has $M_a^*(\gamma) = 0$. Since there is no demand for claims in Period 1, $D(q) = 0$, the equilibrium price is $q_a^*(\pi, 0) = \frac{\alpha_2}{\beta_h}$. Moreover, $V_a^*(\gamma) = \alpha_2$, since all consumers hold the illiquid asset and no trade takes place in Period 1.
For $\beta(\gamma) = \alpha_2$, $\hat{v}^m(\frac{\alpha_2}{\beta_h}; \gamma) = \hat{v}^a(\frac{\alpha_2}{\beta_h}; \gamma)$. Hence, consumers are indifferent about their investment choice. Any aggregate money holdings M_a^* between 0 and $M_a^*(\gamma_o) = \frac{\pi \cdot \alpha_2}{\pi \cdot \alpha_2 + (1-\pi) \cdot \beta_h}$ will be an equilibrium in the market for claims in Period 1. In this case, one has $V_a^*(\gamma) = \hat{v}^m(\frac{\alpha_2}{\beta_h}; \gamma) = \hat{v}^a(\frac{\alpha_2}{\beta_h}; \gamma) = \alpha_2$. This proves case (iii) of the proposition.

Proof of Lemma 1. For a bank choosing reserve policy R in Period 0, the Choquet expected utility of a consumer from a deposit contract is

$$CEU_b(R; \gamma) = [\gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell] \cdot R + [\gamma \cdot \alpha_2 + (1-\gamma) \cdot \beta_\ell \cdot \alpha_1] \cdot (1-R).$$

Proof. Consider the following computations:

$$\begin{aligned} & CEU_b(R; \gamma) \\ = \ & \gamma \cdot [\tau \cdot v_b(\tau; R, h) + (1-\tau) \cdot v_b(\tau; R, \ell)] \cdot p(\tau) \, d\tau + (1-\gamma) \cdot \min_{(t,\tau) \in \{h,\ell\} \times [0,1]} v_b(\tau; R, t) \\ = \ & \gamma \cdot [\pi \cdot v_b(\pi; R, h) + (1-\pi) \cdot v_b(\pi; R, \ell)] + (1-\gamma) \cdot \min_{(t,\tau) \in \{h,\ell\} \times [0,1]} v_b(\tau; R, t) \\ = \ & \gamma \cdot \{\pi \cdot \max\{\beta_h \cdot \rho_1(\pi; R, \pi), \rho_2(\pi; R, \pi)\} + (1-\pi) \max\{\beta_\ell \cdot \rho_1(\pi; R, \pi), \rho_2(\pi; R, \pi)\}\} \\ & + (1-\gamma) \cdot \beta_\ell \cdot [R + \alpha_1 \cdot (1-R)] \\ = \ & \gamma \cdot \{\pi \cdot \beta_h \cdot (1 + i_1(R, \pi)) + (1-\pi) \cdot (1 + i_2(R, \pi)\} + (1-\gamma) \cdot \beta_\ell \cdot [R + \alpha_1 \cdot (1-R)] \\ = \ & [\gamma \cdot \beta_h + (1-\gamma) \cdot \beta_\ell] \cdot R + [\gamma \cdot \alpha_2 + (1-\gamma) \cdot \beta_\ell \cdot \alpha_1] \cdot (1-R). \end{aligned}$$

□

The first equality follows from the definition of Choquet expected utility (Equation (18)). The second equality uses Assumption 2, and the third equality follows from Equation (14). The forth equality is implied by Equations (11) and (12). The last equality uses Equation (10).

Proof of Lemma 3. There is a unique value $\gamma_b \in (0,1)$ for which $V_n^*(\gamma_b) = CEU_b(R^*;\gamma_b)$ holds. Moreover, $\gamma_b > \gamma_o$.

Proof. The critical value γ_b is implicitly defined by the equation $CEU_b(R^*;\gamma_b) - V_n^*(\gamma_b) = 0$. $CEU_b(R^*;\gamma)$ is linear and strictly increasing in γ and $V_n^*(\gamma) = \max\{\alpha_2\,,\, \gamma\cdot\pi\cdot\beta_h + (1-\gamma\cdot\pi)\cdot\beta_\ell\}$ is piecewise linear in γ.
Figure10 illustrates the two possible cases.

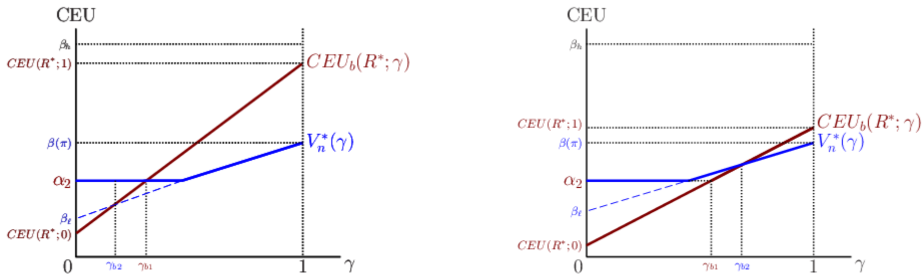


Figure 10: CEU of deposit contact

Denote by γ_{b1} the unique solution of the equation $CEU_b(R^*;\gamma_{b1}) - \alpha_2 = 0$ and by γ_{b2} the also unique solution of $CEU_b(R^*;\gamma_{b2}) - \gamma_{b2}\cdot\pi\cdot\beta_h - (1-\gamma_{b2}\cdot\pi)\cdot\beta_\ell = 0$.
Since $CEU_b(R^*;\gamma)$ is strictly increasing in γ, $\gamma_b = \max\{\gamma_{b1},\gamma_{b2}\}$ holds. The values γ_{b1} and γ_{b2} can be computed explicitly as

$$\begin{aligned}\gamma_b &= \max\{\gamma_{b1},\gamma_{b2}\}\\ &= \max\left\{\frac{(\alpha_2-\beta_\ell)\cdot\alpha_2\cdot\pi + (\alpha_2-\beta_\ell\cdot\alpha_1)\cdot(1-\pi)\cdot\beta_\ell}{(\beta_h-\beta_\ell)\cdot\alpha_2\cdot\pi + (\alpha_2-\beta_\ell\cdot\alpha_1)\cdot(1-\pi)\cdot\beta_\ell},\right.\\ &\qquad\left.\frac{(1-\alpha_1)\cdot\beta_\ell^2}{(\beta_h-\beta_\ell)\cdot(\alpha_2-\beta_\ell)\cdot\pi + \beta_\ell\cdot(\alpha_2-\alpha_1\cdot\beta_\ell)}\right\}.\end{aligned}$$

Straightforward computations show

$$\begin{aligned}\gamma_{b1} - \gamma_o &= \frac{(\alpha_2-\beta_\ell)\cdot\alpha_2\cdot\pi + (\alpha_2-\beta_\ell\cdot\alpha_1)\cdot(1-\pi)\cdot\beta_\ell}{(\beta_h-\beta_\ell)\cdot\alpha_2\cdot\pi + (\alpha_2-\beta_\ell\cdot\alpha_1)\cdot(1-\pi)\cdot\beta_\ell} - \frac{\alpha_2-\beta_\ell}{\beta_h-\beta_\ell}\\ &= \frac{[(\alpha_2-\beta_\ell\cdot\alpha_1)\cdot(1-\pi)\cdot\beta_\ell]\cdot\alpha_2\cdot\pi\cdot(\beta_h-\alpha_2)}{[(\beta_h-\beta_\ell)\cdot\alpha_2\cdot\pi + (\alpha_2-\beta_\ell\cdot\alpha_1)\cdot(1-\pi)\cdot\beta_\ell]\cdot(\beta_h-\beta_\ell)} > 0.\end{aligned}$$

Thus, $\gamma_b = \max\{\gamma_{b1},\gamma_{b2}\} \geq \gamma_{b1} > \gamma_0$. □

Proof of Theorem 1. There exists some level of confidence $\widetilde{\gamma} \in (\gamma_o, 1)$ such that,

(i)	for $\gamma \leq \gamma_o$,	consumers need *no intermediation* at all,
(ii)	for $\gamma_o < \gamma < \widetilde{\gamma}$,	consumers strictly prefer the *secondary asset market*,
(iii)	for $\gamma > \widetilde{\gamma}$,	consumers strictly prefer *bank deposit contracts.*

Proof. We know from Lemma 3 that $\gamma_b > \gamma_o = \frac{\alpha_2 - \beta_\ell}{\beta_h - \beta_\ell}$ holds. Moreover, from Proposition 3 and Equation (17), it is clear that $V_o^*(\gamma_o) = V_a^*(\gamma_o) = \alpha_2$ and $V_o^*(1) = V_b^*(1) > V_a^*(1)$ are true. We will show that (i) V_a^* is a *strictly increasing* and *convex* function of γ on the interval $(\gamma_o, 1)$ and then that (ii) $V_a^*(\gamma_b) > V_b^*(\gamma_b)$ holds.
(i) V_a^* is a *strictly increasing* and *convex* function of γ on $(\gamma_o, 1)$.
We consider first the function $q_a^*(\gamma)$ which is implicitly defined by Equation (24),

$$[\gamma \cdot \pi \cdot \beta_h] \cdot q_a^*(\gamma)^2 + [\alpha_2 - \gamma \cdot \pi \cdot (\alpha_2 + \beta_h) - (1-\gamma) \cdot \beta_\ell)] \cdot q_a^*(\gamma) - [\gamma \cdot (1-\pi) \cdot \alpha_2] \equiv 0.$$

Let $A = \pi \cdot \beta_h$, $B = \alpha_2 - \beta_\ell$, $C = \beta_\ell - \pi \cdot (\alpha_2 + \beta_h)$, $D = -(1-\pi) \cdot \alpha_2$, then this equation may be written equivalently as

$$A \cdot \gamma \cdot q_a^*(\gamma)^2 + [B + C \cdot \gamma] \cdot q_a^*(\gamma) + D \cdot \gamma \equiv 0. \tag{25}$$

By Assumption 1, one has $A > 0$, $B > 0$, $D < 0$ and $A + B + C + D = 0$. Denote by

$$\begin{aligned} \Delta &= 2 \cdot A \cdot \gamma \cdot q_a^*(\gamma) + B + C \cdot \gamma \\ &= A \cdot \gamma \cdot q_a^*(\gamma) - \frac{D \cdot \gamma}{q_a^*(\gamma)} > 0. \end{aligned}$$

The second equality follows from Equation (25), the inequality follows since $\gamma \in [0, 1]$, $q_a^*(\gamma) \in [\frac{\alpha_2}{\beta_h}, \frac{\alpha_2}{\beta_\ell}]$, $A > 0$ and $D < 0$.
Differentiating Equation (25) with respect to γ yields

$$A \cdot q_a^*(\gamma)^2 + 2 \cdot A \cdot \gamma \cdot q_a^*(\gamma) \cdot q_a^{*\prime}(\gamma) + B \cdot q_a^{*\prime}(\gamma) + C \cdot q_a^*(\gamma) + C \cdot \gamma \cdot q_a^{*\prime}(\gamma) + D \equiv 0. \tag{26}$$

Solving for $q_a^{*\prime}(\gamma)$ one obtains

$$\begin{aligned} q_a^{*\prime}(\gamma) &= -\frac{A \cdot q_a^*(\gamma)^2 + C \cdot q_a^*(\gamma) + D}{\Delta} \\ &= \frac{B \cdot q_a^*(\gamma)}{\gamma \cdot \Delta} > 0. \end{aligned} \tag{27}$$

where the second equality uses again Equation (25). The inequality follows since $B > 0$. This establishes that $q_a^*(\gamma)$ is a strictly increasing function of γ.
Differentiating $V_a^*(\gamma)$, as in Equation (9),

$$V_a^*(\gamma) = \gamma \cdot \pi \cdot q_a^*(\gamma) \cdot \beta_h + (1 - \gamma \cdot \pi) \cdot \alpha_2,$$

with respect to γ yields

$$\begin{aligned}\frac{\partial V_a^*(\gamma)}{\partial \gamma} &= \pi \cdot (q_a^*(\gamma) \cdot \beta_h - \alpha_2) + \gamma \cdot \pi \cdot \beta_h \cdot q_a^{*\prime}(\gamma) \\ &\geq \pi \cdot \left(\frac{\alpha_2}{\beta_h} \cdot \beta_h - \alpha_2\right) + \gamma \cdot \pi \cdot \beta_h \cdot q_a^{*\prime}(\gamma) \\ &= \gamma \cdot \pi \cdot \beta_h \cdot q_a^{*\prime}(\gamma) > 0.\end{aligned} \tag{28}$$

The weak inequality follows from $q_a^*(\gamma) \geq \frac{\alpha_2}{\beta_h}$, the strict inequality from Equation (27). Hence, $V_a^*(\gamma)$ is a *strictly increasing* function of γ.
Differentiating Equation (28) again with respect to γ yields

$$\frac{\partial^2 V_a^*(\gamma)}{\partial \gamma^2} = \pi \cdot \beta_h \cdot \left[2 \cdot q_a^{*\prime}(\gamma) + \gamma \cdot q_a^{*\prime\prime}(\gamma)\right] \tag{29}$$

which is positive if $2 \cdot q_a^{*\prime}(\gamma) + \gamma \cdot q_a^{*\prime\prime}(\gamma) > 0$ holds. In order to establish this inequality, we differentiate the identity in Equation (26) with respect to γ,

$$q_a^{*\prime\prime}(\gamma) \cdot \Delta + \left[4 \cdot A \cdot \gamma \cdot q_a^*(\gamma) \cdot q_a^{*\prime}(\gamma) + 2 \cdot A \cdot \gamma \cdot q_a^{*\prime}(\gamma)^2 + 2 \cdot C \cdot q_a^{*\prime}(\gamma)\right] \equiv 0.$$

Solving for $q_a^{*\prime\prime}(\gamma)$, we have

$$q_a^{*\prime\prime}(\gamma) = -\frac{1}{\Delta} \cdot \left[4 \cdot A \cdot \gamma \cdot q_a^*(\gamma) \cdot q_a^{*\prime}(\gamma) + 2 \cdot A \cdot \gamma \cdot q_a^{*\prime}(\gamma)^2 + 2 \cdot C \cdot q_a^{*\prime}(\gamma)\right]. \tag{30}$$

Hence,

$$\begin{aligned}&2 \cdot q_a^{*\prime}(\gamma) + \gamma \cdot q_a^{*\prime\prime}(\gamma) \\ &= \frac{1}{\Delta} \cdot \left[2 \cdot \Delta \cdot q_a^{*\prime}(\gamma) - \gamma \cdot \left(4 \cdot A \cdot \gamma \cdot q_a^*(\gamma) \cdot q_a^{*\prime}(\gamma) + 2 \cdot A \cdot \gamma \cdot q_a^{*\prime}(\gamma)^2 + 2 \cdot C \cdot q_a^{*\prime}(\gamma)\right)\right] \\ &= \frac{2 \cdot q_a^{*\prime}(\gamma)}{\Delta} \cdot \left[B - A \cdot \gamma^2 \cdot q_a^{*\prime}(\gamma)\right] \\ &= \frac{2 \cdot q_a^{*\prime}(\gamma)}{\Delta} \cdot \left[B - A \cdot \gamma^2 \cdot \frac{B \cdot q_a^*(\gamma)}{\gamma \cdot \Delta}\right] \\ &= \frac{2 \cdot B \cdot q_a^{*\prime}(\gamma)}{\Delta^2} \cdot \left[A \cdot \gamma \cdot q_a^*(\gamma) + B + C \cdot \gamma\right] \\ &= \frac{2 \cdot B \cdot q_a^{*\prime}(\gamma)}{\Delta^2} \cdot \left[-\frac{D \cdot \gamma}{q_a^*(\gamma)}\right] > 0\end{aligned}$$

where the first equality uses Equation (30), the second equality follows by straightforward computations, the third equality uses Equation (27) and the forth equality follows again from straightforward computations. The final equality uses the identity in Equation (25).
In combination with the second-order derivative of $V_a^*(\gamma)$ in Equation (29) this establishes *convexity* of the function $V_a^*(\gamma)$.
(ii) $V_a^*(\gamma_b) > V_b^*(\gamma_b)$.
We consider two cases: $\gamma_b = \gamma_{b1} > \gamma_{b2}$ and $\gamma_b = \gamma_{b2} > \gamma_{b1}$.
Case (i): Suppose $\gamma_b = \gamma_{b1}$. By definition of γ_{b1}, $V_b^*(\gamma_{b1}) = CEU_b(R^*; \gamma_{b1}) = \alpha_2$. Since V_a^* is a *strictly increasing* function on $(\gamma_o, 1)$, and $\gamma_{b1} > \gamma_o$ by

Lemma 3, we have $V_a^*(\gamma_b) > V_a^*(\gamma_o) = \alpha_2 = V_b^*(\gamma_{b1}) = V_b^*(\gamma_b)$, where the last equation follows by the hypothesis $\gamma_b = \gamma_{b1}$.

Case (ii): Suppose now $\gamma_b = \gamma_{b2} > \gamma_{b1} > \gamma_o$. As V_a^* is a strictly increasing and strictly convex function on $(\gamma_o, 1)$ and both illiquid asset and money are held, we have $V_a(\gamma) > V_n(\gamma)$ for all $\gamma \in (\gamma_o, 1]$. So by definition of γ_{b2} we have $V_a^*(\gamma_{b2}) > V_b^*(\gamma_{b2})$.

Since $V_b^*(1) > V_a^*(1)$ and $V_b^*(\gamma_b) < V_a^*(\gamma_b)$ hold, V_a^* is a *strictly increasing* and *convex* function of γ on $(\gamma_o, 1)$, and $V_b^*(\gamma)$ is a strictly increasing and linear function of γ on $(\gamma_b, 1)$, by the intermediate value theorem, there must be a unique $\widetilde{\gamma} \in (\gamma_b, 1)$ such that $V_a^*(\widetilde{\gamma}) = V_b^*(\widetilde{\gamma})$ holds. For values of $\gamma \in (\gamma_o, \widetilde{\gamma})$, we have $V_a^*(\gamma) > V_b^*(\gamma)$, for values $\gamma \in (\widetilde{\gamma}, 1]$, we have $V_b^*(\gamma) > V_a^*(\gamma)$, and for $\gamma \in [0, \gamma_o)$, $V_a^*(\gamma) = V_b^*(\gamma) = \alpha_2$. □