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DOUBLE DEGENERACY IN THE IMMISCIBLE WILTON RIPPLE PHENOMENON: A THREE PERTURBATION PARAMETER PROBLEM IN BIFURCATION THEORY

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Abstract A study is made of the waves which may occur at the interface of two fluids and which arise from an interaction between the two lowest adjacent harmonics of the motion. The problem is studied by reformulating it, via a conformal mapping transformation, into a pair of coupled non-linear functional differential equations which are invariant under the action of the group $O(2)$. The classical method of Lyapunov-Schmidt is then employed to replace this formulation by a pair of algebraic equations, known as the bifurcation equations. Throughout our analysis we impose a physical condition which essentially means that the densities and phase speeds in each fluid are approximately equal. This means that the bifurcation equations assume a cubic form rather than the quadratic form which they generically possess in problems of this type. The bifurcation equations are then analysed by means of singularity theory so that we are able to draw the solution curves of the system. A large variety of solution diagrams are found exhibiting such behaviour as multiple primary bifurcation, secondary bifurcation and bifurcation from infinity.

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1. Introduction

This report makes a study of the small amplitude ripples which arise on the horizontal interface of two ideal fluids and which are formed by an interaction between the first two harmonics of the motion. Such waves are now known, perhaps not entirely accurately, as Wilton ripples, see [Harrison 1909, Wilton 1915]. We shall study this problem by using the hodograph transformation to cast it as a pair of nonlinear functional equations on the unit disc. This formulation seems to be due to Kotchine [1928], although it was re-derived much more recently by Okamoto [1990]. The unknowns in the equations are θ and U where θ represents the angle of inclination between the disturbed interface and the horizontal and U connects the parameterisations in the two fluids. There are also three parameters in the problem: two are the fluid speeds and the third is surface tension. Clearly whatever values are assigned to the parameters, $\theta = U = 0$ is always a solution, representing a horizontal interface. However, at certain distinguished values of the parameters the null-space of the linearised problem is spanned by $\sin s$ and $\sin 2s$, such points on the line of trivial solutions are potential *bifurcation points* at which a curve of nontrivial solutions may (or may not) branch off from the trivial set. If the parameters are perturbed slightly from their distinguished values, then the double null-space splits into two single null-spaces. This case is then covered by the classical "bifurcation from a simple eigenvalue" which guarantees the existence of a single bifurcation curve. In order to determine the configuration of any solution curves, we employ the classical method of Lyapunov-Schmidt by which means the original infinite dimensional problem may be reduced to a finite set of algebraic equations, known as the *bifurcation equations.*

All our analysis will be concerned with the particular case when $\rho_1 c_1^2 \sim \rho_2 c_2^2$, where ρ_i and c_i represent the densities and phase speeds of the respective fluids. The reason for imposing this condition, apart from its obvious physical importance, is that the bifurcation equations turn out to be cubic in nature, rather than the quadratic form which they generically assume. The details of the mathematical analysis are therefore completely different. For a treatment of the case when this condition is relaxed, see [Jones 2015]. Having found the bifurcation equations, we then employ techniques from singularity theory to analyse them. The most important result is to determine their unfoldings whereby we are able to incorporate any perturbation of the problem by a judicious choice of a finite number of additional parameters. In this way we are able to account both for detuning in the 1:2 resonance and also imperfections in the phase speed condition given above. Such situations have obvious experimental ramifications.

Having analysed the equations, we are then in a position to draw the bifurcation diagrams. Full details are given in §5 where we find that a large variety of solution configurations are possible.

Both single and multiple bifurcations may occur from the line of trivial solutions and further secondary bifurcations may occur along these curves. The solution curves may be bounded or unbounded.

As already mentioned, the first discussions of the Wilton ripple phenomenon in the literature occurred over one hundred years ago but most of the work concerning them seems to have been undertaken within the last fifty years. The first such study seems to have been that of Pierson and Fife [1961] who employed perturbation methods to study Wilton ripples on a free surface of a fluid of infinite depth. This work was generalised by a number of researchers [McGoldrick 1965, 1970, 1972, Nayfeh 1970, 1972, 1973] to allow for such effects as finite depth, modulations in the frequency, more general resonances and interactions between different media but the first truly rigorous investigation was that of Reeder and Shinbrot [1981] who used sophisticated analytic techniques to prove the existence of these waves. A comprehensive, but formal study of resonant surface waves was conducted by Chen and Saffman [1979] and later Jones and Toland [1985,1986] made a rigorous study of the same problem, describing the results in the context of bifurcation theory. They cast the problem as an integral equation but later Okamoto [1990] used a somewhat simpler differential equation. This formulation was later used by Okamoto and Shoji [1996] who tracked the bifurcation curves by numerical means and also by Jones [1996,1997] who employed an analytical approach.

2. Kotchine's equation

We shall be concerned with the waves which arise at the interface of two stratified ideal fluids of infinite vertical extent. When the system is in its undisturbed state the interface is horizontal and we shall take this to be the x-axis of a standard two-dimensional Cartesian co-ordinate system. The lower fluid flows horizontally with velocity c_1 and the upper fluid with velocity c_2. The fluids have densities ρ_1 and ρ_2 respectively where the lighter fluid is taken to be on top so that the ratio ρ=ρ_2/ρ_1 lies between zero and unity. The remaining parameters in the problem are g, the force of gravity, which acts in the negative y-direction and S which represents capillary effects at the interface. The motion is taken to be irrotational and periodic so that one period occupies the region

$R = \{(x, y) : -\pi < x < \pi, -\infty < y < \infty\}$.In addition the configuration of the interface is given by $y = H(x)$ where $H(x)$ is a 2π periodic function (*a priori* unknown, of course). The physical situation is depicted in Fig.1.

Shortly we present a formulation of this problem as an integro-differential equation originally due to Kotchine [1928], for a more recent derivation see Okamoto [1997]. Before giving the equation we briefly record some facts concerning function spaces. For a nonnegative integer r

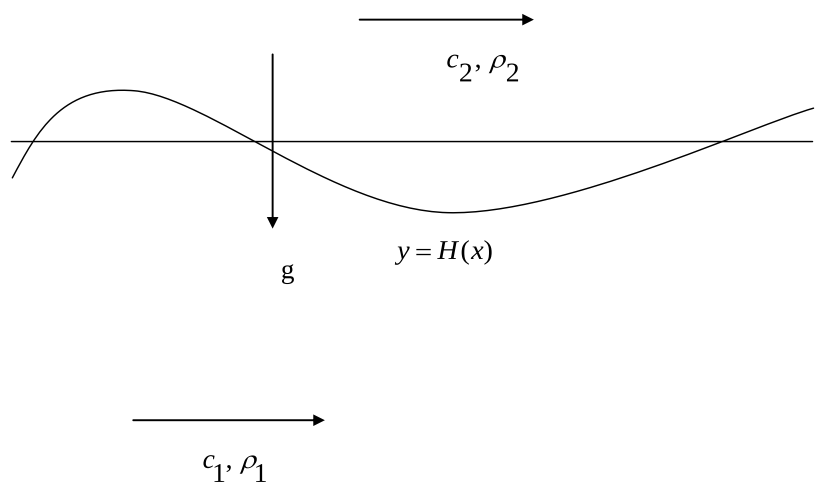


Fig 1

let C^r denote the Banach space of real valued 2π periodic r-times continuously differentiable functions whose Fourier series contain no constant term. Set, for $n = 1$ or 2,

$$X_n^r = \{f \in C^r : f(s) = f(s + 2\pi/n)\} \tag{1}$$

Then we define a nonlinear operator H (the Hilbert transform, see [Okamoto & Shoji, 1996]) on X_n^2 by

$$H(\sum_1^\infty a_k e^{iks} + \bar{a}_k e^{-iks}) = \sum_1^\infty -ia_k e^{iks} + i\bar{a}_k e^{-iks}. \tag{2}$$

We can now state our first main result which enables us to connect solutions of the immiscible fluid problem to that of an integro-differential equation.

THEOREM 2.1 Let $(\theta, U) \in X_1^2 \times X_1^2$ be such that $|\theta| < \pi/2$ and $s + U(s)$ is a bijection on $[-\pi, \pi]$. Further, define τ as $H\theta$ and τ^* as $H(\theta(I + U)^{-1})(s + U(s))$. Let p, q and b be real numbers. Now set

$$F_1 = \frac{1}{2}\frac{d}{ds}\Big\{\exp 2\tau - b\exp(-2\tau^*)\Big\} - p\exp(-\tau)\sin\theta + q\frac{d}{ds}\left(\exp\tau\frac{d\theta}{ds}\right) \tag{3a}$$

and

$$F_2 = \frac{dU}{ds} - \exp(-\tau - \tau^*) + 1. \tag{3b}$$

Then

(a) $F = (F_1, F_2)$ is a well-defined operator from $X_1^2 \times X_1^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to $X_1^0 \times X_1^0$.

(b) $F_i(X_2^2, X_2^2, ., ., .) \subseteq X_2^0$, i=1,2.

(c) If $F(\theta, U, p, q, b) = 0$,then there is a solution to the immiscible capillary-gravity wave problem. The dimensionless parameters are related to the physical quantities in the following way

$b = \rho c_2^2 / c_1^2, \quad p = g(1-\rho)/c_1^2 \quad \text{and} \quad q = S/\rho_1 c_1^2.$

(d) The angle of the disturbance is represented by θ so that $dH/dx = \tan\theta$. Further, the interface is parameterised as

$$(x(s), y(s)) = \left(-\int_0^s e^{-\tau(t)} \cos\theta(t) dt, \quad -\int_0^s e^{-\tau(t)} \sin\theta(t) dt \right).$$

A proof of this result is to be found in [Kotchine 1928, Okamoto 1997] or alternatively follows by a routine modification of the analogous result in [Jones 1997, Okamoto 1990] for a single fluid.

REMARK The operator F is meaningful for all real p, q and b. However, in order to be physically relevant these quantities must all be positive.

Since we are dealing with small amplitude waves our first step is to linearise the equations around the zero solution. They become

$$\frac{(1+b)\tau'}{3} - p\theta + q\frac{\theta''}{3} = 0 \tag{4a}$$

and

$$\frac{dU}{ds} + 2\tau = 0. \tag{4b}$$

We seek solutions formed by the interaction of a fundamental and its second harmonic. An easy calculation yields that for any value of b, if $p = \frac{2(1+b)}{3}$ and $q = \frac{1+b}{3}$ then

$$(\theta, U) = (e^{\pm is}, 2e^{\pm is}), (e^{\pm 2is}, e^{\pm 2is}). \tag{5}$$

are the solutions to (4). However it is our intention to study the solutions which occur when b is equal or close to unity. The reasons for this are both mathematical and physical. Physically, the case $b = 1$ corresponds to $\rho_1 c_1^2 = \rho_2 c_2^2$ which is a case of obvious physical importance. Mathematically, as will shortly become clear, this case is of especial interest because the bifurcation equations exhibit non-generic properties, being cubic rather than quadratic in nature. Let us fix some notation. In order to allow for perturbations in the problem, we introduce three parameters α, β and γ. Then we set $b = 1 + 3\gamma$ (the coefficient 3 makes the calculations slightly more tractable), so that $p = 4/3 + \alpha + 2\gamma$ and $q = 2/3 + 2\gamma + \beta$ and regard the operator F as a function of these new variables. The next step in the analysis is to decompose the space X_1^2 as $\mathrm{sp}\{e^{\pm is}, e^{\pm 2is}\} \oplus Y$, (where Y is the complementary subspace) and write a solution to the system as $\theta = e_1 + f_1$ and $U = e_2 + f_2$ where $f_i \in Y$ and

$$e_1 = -ize^{2is} - iwe^{is} + cc, e_2 = -ize^{2is} - 2iwe^{is} + cc, (z, w) \in \mathbb{C}^2, \tag{6}$$

where cc stands for complex conjugate.

Further let P be the projection of X_1^0 onto $\mathrm{sp}\{e^{\pm is}, e^{\pm 2is}\}$, and let $Q = I - P$. It therefore follows that $F = 0$ is equivalent to the equations

$$PF_i(e_1 + f_1, e_2 + f_2, \alpha, \beta, \gamma) = 0 \tag{7a}$$

$$QF_i(e_1 + f_1, e_2 + f_2, \alpha, \beta, \gamma) = 0, \quad i = 1, 2. \tag{7b}$$

Classically, the Implicit Function Theorem then allows us to assert that (7b) has a solution $f_i(e_1, e_2, \alpha, \beta, \gamma,)(i = 1, 2)$ for $(e_1, e_2, \alpha, \beta, \gamma)$ in a neighbourhood of the origin in $X_1^2 \times X_1^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. To proceed with the analysis we need the f_i to quadratic order. A straightforward Taylor expansion of F_i leads us to the following equations

$$\left\{\frac{2}{3}f_1 - \frac{1}{3}f_1'' - \tilde{f}_1'\right\} = \frac{d}{ds}\left\{\frac{1}{2}e_2\tilde{e}_1' - \frac{1}{2}H(e_2e_1') + \frac{1}{3}e_1'\tilde{e}_1\right\} + \frac{2}{3}(\tilde{e}_1e_1) \tag{8a}$$

and that

$$\frac{df_2}{ds} = H(e_2e_1') - e_2\tilde{e}_1' + 2\tilde{e}_1^2 \tag{8b}$$

where the tilde stands for conjugate and the terms on the right hand sides are to be projected onto Y.

Use of these equations then leads to the following expressions for f_i

$$f_1 = -iy^2 e^{4is} - \frac{11i}{4} wye^{3is} + cc \tag{9a}$$

$$f_2 = -\frac{i}{2} y^2 e^{4is} - \frac{4i}{3} wye^{3is} + cc. \tag{9b}$$

In order to determine the bifurcation equations we need to expand the operator F_1 to cubic order. It turns out that

$$F_1 = 2\tau' - (4/3)\theta + (2/3)\theta'' - \alpha\theta + \beta\theta'' + \gamma\theta'' +$$

$$3\gamma\tau' - 2\gamma\theta + \frac{d}{ds}\{U\tau' - H(U\theta') + (2/3)\tau\theta'\} +$$

$$(4/3)\tau\theta + \alpha\tau\theta + \beta(\tau\theta')' +$$

$$\gamma\frac{d}{ds}\{3U\tau' - 3H(U\theta') - 3\tau^2 + \tau\theta'\} + 2\gamma\tau\theta +$$

$$\frac{d}{ds}\{(4/3)\tau^3 + (1/2)U^2\tau'' - U(H(U\theta'))' +$$

$$H(U^2\theta'')/2 - 2U\tau\tau' + 2\tau H(U\theta') + (1/3)\tau^2\theta'\} +$$

$$(2/9)\theta^3 - (2/3)\tau^2\theta. \tag{10}$$

A lengthy calculation, making use of MATHEMATICA, now yields the algebraic form of the bifurcation equationsis

$$\alpha y + 4\beta y + \alpha w^2 - 2\beta w^2 - 6\gamma w^2 + 8|y|^2 y + \frac{16}{3}|w|^2 y + h(y, w, \alpha, \beta, \gamma) = 0, \qquad (11a)$$

$$\alpha w + \beta w - 3\beta y\overline{w} - 3\gamma y\overline{w} + \frac{31}{3}|y|^2 w + 5|w|^2 w + wk(y, w, \alpha, \beta, \gamma) = 0. \qquad (11b)$$

Equations (11) represent the reduction of the original infinite dimensional problem.

REMARKS 1. Equation (11a) is the projection onto e^{2is} while (11b) is the projection onto e^{is}.

2. The remainder terms satisfy $h = O(|(y, w, \alpha, \beta, \gamma)|^4)$ and $k = O(|(y, w, \alpha, \beta, \gamma)|^3)$ as $(y, w, \alpha, \beta, \gamma) \to 0$. The reason for the special form of the second remainder is the invariance property noticed in Thm 2.1(b). For we see from this that it would be perfectly possible carry out the reduction procedure in X_2^r (rather than in X_1^r) in the context of which the null space of the linearized operator is spanned by the single harmonic e^{2is}. It therefore follows that the left hand side of (11b) must vanish along the ray $w = 0$.

3. It should be particularly noted that the bifurcation equations are typically cubic because the quadratic terms w^2 and $y\overline{w}$ only arise through the parameters. This is in contrast to the case when $b \neq 1$ when both the quadratic terms are present with coefficients proportional to $1 - b$ (for this case see [Okamoto, 1997]). It is this vanishing of *both* the quadratic terms which renders this situation especially remarkable and justifies our description of double degeneracy.

3. Some equivariance results

In this section we gather together some algebraic results which enable us to analyse the structure of the bifurcation equations. For treatments of bifurcation equations enjoying similar symmetries, see [Armbruster & Dangelmayr 1986, 1987], [Buzano & Russo 1987], [Dangelmayr 1986], [Jones 1996].

Let $O(2)$ denote the orthogonal group, which then induces an action on X_n^r whose generators are

$$Sf(s) = -f(-s) \quad \text{and} \quad R_\lambda f(s) = f(s+\lambda), 0 \leq \lambda < 2\pi. \tag{12}$$

Then we have (see [Okamoto 1990, 1997])

THEOREM 3.1 The operator F is equivariant under the group action of $O(2)$, that is

$$F_i(\omega\theta, \omega U, \cdot, \cdot, \cdot) = \omega F_i(\theta, U, \cdot, \cdot, \cdot), \quad i = 1, 2 \text{ for all } \omega \in O(2). \tag{13}$$

Let Γ be a topological group which acts on a vector space V.

DEFINITION 3.1(a) Suppose $f : V \to \mathbb{R}$ satisfies $f(\omega v) = f(v)$ for all $\omega \in \Gamma, v \in V$. Then we say that f is an *invariant* function. The set of invariant functions forms ring over $\mathbb{R}$.

(b) Suppose $f : V \to V$ satisfies $f(\omega v) = \omega f(v)$ for all $\omega \in \Gamma, v \in V$. Then we say that f is an *equivariant* function. The equivariant functions form a module over the invariant functions.

DEFINITION 3.2(a) A set $u_1, \ldots, u_s$ of invariant functions is said to be a *generating set* if every invariant function f can be expressed as $f = \tilde{f}(u_1 \ldots u_s)$, for some $\tilde{f} : \mathbb{R}^s \to \mathbb{R}$.

(b) A set $g_1 \ldots g_r$ of equivariant functions is said to be a *generating set* if every equivariant function g may be written as $g = f_1 g_1 + \ldots + f_r g_r$ where the f_i are invariant functions.

Now recall that we have been able, via the Lyapunov-Schmidt procedure, to reduce the problem from one cast in the space X_1^2 to one in the space $\mathrm{sp}\{e^{is}, e^{2is}\}$, which we identify with $\mathbb{C}^2$. Thus the group action of $O(2)$ noticed in Thm 3.1 now restricts to $\mathbb{C}^2$ and has the generators

$$\kappa(y, w) = (\overline{y}, \overline{w}) \quad \text{and} \quad \lambda(y, w) = (e^{2i\lambda} y, e^{i\lambda} w), 0 \leq \lambda < 2\pi. \tag{14}$$

Standard equivariant group theory [Armbruster and Dangelmayr 1986, 1987], [Jones 1996] then yields that the generators for the invariant functions are $|y|^2, |w|^2$ and $y\overline{w}^2 + \overline{y} w^2$ while those for the equivariant functions are

$$(y, 0), \quad (0, w), \quad (w^2, 0), \quad (0, y\overline{w}). \tag{15}$$

Clearly then, the bifurcation equations (11) are forced to take the generic form

$$y f_1 + w^2 f_2 = 0 \tag{16a}$$

$$w f_3 + y\overline{w} f_4 = 0 \tag{16b}$$

where $f_i \equiv f_i(|y|^2, |w|^2, y\overline{w}^2 + \overline{y}w^2)$.

Note that in this particular case $f_i(0,0,0) = 0$ for both $i = 2$ and $i = 4$ thus again emphasising the double degeneracy in the problem. The system (16) may be considerably simplified by employing the polars $y = xe^{i\phi}, w = ye^{i\psi}$, which means that it becomes

$$xf_1 + \cos(2\psi - \phi)y^2 f_2 = 0 \tag{17a}$$

$$\sin(2\psi - \phi)y^2 f_2 = 0 \tag{17b}$$

$$yf_3 + \cos(2\psi - \phi)xyf_4 = 0. \tag{17c}$$

Clearly $\sin(2\psi - \phi) = 0$ and so the other equations become

$$xf_1 \pm y^2 f_2 = 0 \qquad yf_3 \pm xyf_4 = 0 \tag{18$^\pm$}$$

where $f_i \equiv f_i(x^2, y^2, \pm 2xy^2)$. It is now easy to see that if (x, y) satisfies $(18)^+$, then $(-x, y)$ is a solution of $(18)^-$ and hence we may confine our attention to consideration of $(18)^+$.

Thus we see that the unknowns in the bifurcation equations may be restricted to real values. However even more is true, for in this context the group action is now one of $\mathbb{Z}_2$ on $\mathbb{R}^2$ and is given by $(x, y) \to (x, -y)$ so that the invariant functions are generated by x and y^2 and the equivariant functions are generated by $(1, 0)$ and $(0, y)$.

4. Unfolding the equations

In this section which, together with the next, contains the main results of this paper we use methods from algebraic topology to discuss the bifurcation equations. As a preliminary step we shall set the perturbation parameters β and γ to zero, later we shall see how they may be subsumed within a more general context. Bearing in mind the results of the previous section, we see that the equations may be written as

$$\alpha x + \alpha y^2 + 8x^3 + \frac{16}{3}xy^2 + p_1 = 0, \quad \alpha y + \frac{31}{3}x^2 y + 5y^3 + yp_2 = 0. \tag{19}$$

We shall abbreviate the left hand sides as $g_1 + p_1, yg_2 + yp_2$ (or simply $g + p$) and sometimes write v for the invariant coordinate y^2. Now we need some concepts from algebraic geometry. We only state the particular forms of the results which we need, for proofs and generalisations see [Golubitsky & Schaeffer 1979, 1985, 1988].

DEFINITION 4.1(a) Let $\mathcal{E}(\mathbb{Z}_2)$ be the ring (over $\mathbb{R}$) consisting of real-valued $\mathbb{Z}_2$ invariant functions from $\mathbb{R}^2 \times \mathbb{R}$ to $\mathbb{R}$. A typical member then has the form $f(x, v, \alpha) \equiv f(x, y^2, \alpha)$. We shall use $< x, v >$ to denote the subring generated by x and v, similarly for $< v, \alpha >$ and $< \alpha >$ etc. The expression $< x, v >^2$ denotes the subring of functions of the form $f \cdot g$ where $f, g \in < x, v >$, similarly for higher powers. In addition, we shall sometimes use the notation $\mathcal{M}$ for $< x, v, \alpha >$, this is sometimes called the the *maximal ideal* of $\mathcal{E}(\mathbb{Z}_2)$.

(b) Let $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ denote the module (over $\mathcal{E}(\mathbb{Z}_2)$) of $\mathbb{Z}_2$-equivariant functions from $\mathbb{R}^2 \times \mathbb{R}$ to $\mathbb{R}^2$. A typical member thus has the form (f_1, yf_2) where $f_i \in \mathcal{E}(\mathbb{Z}_2)$, we shall often write this as $[f_1, f_2]$. In addition, let $\overrightarrow{\mathcal{M}}(\mathbb{Z}_2)$ denote the submodule consisting of those functions which vanish at the origin. It is therefore clear that $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2) = \overrightarrow{\mathcal{M}}(\mathbb{Z}_2) \oplus \mathbb{R}\{Y_1\}$ where $Y_1 = (0, y)$ (alternatively [0,1]). The following result is somewhat technical but will be needed later on in the proof of Thm 4.4, for a proof consult [Golubitsky and Schaeffer 1985].

LEMMA 4.1 (Nakayama's lemma) Let $\overrightarrow{\mathcal{I}}$ and $\overrightarrow{\mathcal{J}}$ be submodules of $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$. Then $\overrightarrow{\mathcal{I}} \subseteq \overrightarrow{\mathcal{J}}$ if and only if $\overrightarrow{\mathcal{I}} \subseteq \overrightarrow{\mathcal{J}} + \mathcal{M}.\overrightarrow{\mathcal{J}}$.

DEFINITION 4.2 We define $\overleftrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ to be the module over $\mathcal{E}(\mathbb{Z}_2)$ consisting of all linear maps $S((x,y),\alpha) : \mathbb{R}^3 \to \mathbb{R}^2$ which satisfy

$$S(\omega(x,y),\alpha)\omega = \omega S((x,y),\alpha) \quad \text{for all } (x,y) \in \mathbb{R}^2, \omega \in \mathbb{Z}_2, \alpha \in \mathbb{R}.$$

DEFINITION 4.3 If $f, g \in \overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$, then f and g are said to be *strongly* $\mathbb{Z}_2$*-equivalent* if there exists an invertible change of co-ordinates

$$((x,y),\alpha) \to (\hat{x}((x,y),\alpha), \hat{y}((x,y),\alpha), \alpha) \quad \text{and} \quad S((x,y),\alpha) \in \overleftrightarrow{\mathcal{E}}(\mathbb{Z}_2)$$

such that

$$g((x,y),\alpha) = S((x,y),\alpha)f((\hat{x},\hat{y}),\alpha)$$

and $(\hat{x},\hat{y})$ vanish at the origin in $\mathbb{R}^3$.

Loosely speaking, this is an important concept because if two maps are strongly $\mathbb{Z}_2$-equivalent then their zero sets are qualitatively the same. The next result is also of great importance. It will enable us to determine when two maps are strongly $\mathbb{Z}_2$-equivalent.

DEFINITION 4.4 Let $f \in \overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$.

(a) The *restricted tangent space* of f, written $RT(f)$, is the space generated by $S_i f$ and $(df)X_j$ where the S_i generate $\overleftrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ and the X_j generate $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$.

(b) The *tangent space* of f, denoted by $T(f)$, is defined to be to be

$$T(f) = RT(f) \oplus \mathbb{R}\{(df)Y_1, f_\alpha, \alpha f_\alpha, \alpha^2 f_\alpha \ldots\}$$

where Y_1 is as defined in 4.1(b).

Note that both $RT(f)$ and $T(f)$ are submodules of $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$.

Now we come to another important result, a proof may be found in [Golubitsky & Schaeffer 1985].

THEOREM 4.2 Let f and $p \in \overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ such that

$$RT(f+tp) = RT(f)$$

for all $t \in [0,1]$. Then $f+tp$ is strongly $\mathbb{Z}_2$ equivalent to f for all $t \in [0,1]$.

The importance of this result is that it enables us to justify our neglect of the higher order terms in the bifurcation equations. Next we introduce the concept of unfoldings. This is needed because it allows us to incorporate perturbations into the bifurcation equations.

DEFINITION 4.5 Let $f \in \overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$. Then a *k-parameter* $\mathbb{Z}_2$ *-unfolding* of f is a map F from $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^k$ to $\mathbb{R}^2$ such that

$$F((x,y),\alpha,0) = f((x,y),\alpha)$$

$$\text{and} \quad F(\omega(x,y),\alpha,\delta) = \omega F((x,y),\alpha,\delta)$$

for all $\omega \in \mathbb{Z}_2$ and $((x,y),\alpha,\delta) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^k$. Now let $\hat{F}((x,y),\alpha,\hat{\delta})$ be an ℓ-parameter $\mathbb{Z}_2$-unfolding of f. We say $\hat{F}$ *factors through* F if

$$\hat{F}((x,y),\alpha,\hat{\delta}) = S((x,y),\alpha,\hat{\delta})F(X((x,y),\alpha,\hat{\delta}),\Lambda(\alpha,\hat{\delta}),A(\hat{\delta}))$$

where

(a) $S \in \overleftrightarrow{\mathcal{E}}(\mathbb{Z}_2)$, $S((x,y),\alpha,0) = I$ and $S(\omega(x,y),.,.)\omega = \omega S((x,y),.,.)$

(b) $X\colon \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\ell \to \mathbb{R}^2$ is such that

$$X((x,y),\alpha,0) = (x,y) \text{ and } X(\omega(x,y),\alpha,\hat{\delta}) = \omega X((x,y),\alpha,\hat{\delta}),$$

(c) $\Lambda(\alpha, 0) = \alpha$ and $A(0) = 0$.

DEFINITION 4.6 A $\mathbb{Z}_2$-unfolding F of f is said to be *universal* if

(a) every $\mathbb{Z}_2$-unfolding $\hat{F}$ of f factors through F,

(b) F depends on the minimum number of parameters needed for condition (a) to be satisfied.

The minimum number appearing in the definition of a universal unfolding is called the *codimension* of f. It is a remarkable fact that in general the codimension of f is finite: in other words *all* possible perturbations of f may be described by adjoining a *finite* number of fresh parameters. The following result presents us with a procedure by which we may determine the additional parameters required for a universal unfolding.

THEOREM 4.3 Let $f \in \overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ and let $W \subseteq \overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ be the subspace such that

$$\overrightarrow{\mathcal{E}}(\mathbb{Z}_2) = T(f) + W.$$

Let $p_1((x,y),\alpha) \dots p_k((x,y),\alpha)$ be a basis for W. Then

$$F((x,y),\alpha,\delta) = f((x,y),\alpha) + \sum_{j=1}^{k} \delta_j p_j((x,y),\alpha)$$

is a universal $\mathbb{Z}_2$-unfolding of f.

We now come to our first major result.

THEOREM 4.4 The expression $g+p$ is strongly $\mathbb{Z}_2$ equivalent to g. In fact

$$RT(g+p) = RT(g) = \mathcal{W} + [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] \tag{20}$$

where $\mathcal{M} =< \alpha, x, v >$ and $\mathcal{W} = \mathbb{R}\{q_i, 1 \leq i \leq 7\}$ where

$$q_1 = [\alpha v, 0], \quad q_2 = [v^2, 0], \quad q_3 = [\alpha^2, 0], \quad q_4 = [3\alpha x + 16xv + 24x^3, 0],$$

$$q_5 = [0, 3\alpha + 15v + 31x^2], \quad q_6 = [48x^3, 62x^2], \quad q_7 = [16xv, 15v]. \tag{21}$$

PROOF Throughout the proof, we shall write $f_i = g_i + tp_i$ where $t \in [0,1]$. The first thing to observe is that by [Golubitsky *et al*, 1988, p177] $RT(f)$ is the submodule of $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$ with the eight generators

$$[f_1, 0], [vf_2, 0], [0, f_1], [0, f_2], [xf_{1x}, xf_{2x}], [vf_{1x}, vf_{2x}], [\alpha f_{1x}, \alpha f_{2x}], [vf_{1v}, vf_{2v}]. \tag{22}$$

The next stage is to show that

$$[< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] \subseteq RT(f). \tag{23}$$

In order to show this, let $\mathcal{J} \subseteq RT(f)$ be the submodule with the twenty-two generators

$$u[f_1, 0], u[vf_2, 0], [0, f_1], u[0, f_2],$$

$$u[xf_{1x}, xf_{2x}], u[vf_{1x}, vf_{2x}], u[\alpha f_{1x}, \alpha f_{2x}], u[vf_{1v}, vf_{2v}], \tag{24}$$

where $u = x, v, \alpha$.

We now claim that

$$\mathcal{J} = [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] \tag{25}$$

which will imply the truth of (23). We shall prove our claim at the end but let us proceed with the description of $RT(f)$. In fact we can now show that

$$RT(f) = [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] + \mathcal{W} \tag{26a}$$

where

$$\mathcal{W} = \mathbb{R}\{[f_1, 0], [vf_2, 0], [0, f_2], [xf_{1x}, xf_{2x}], [vf_{1x}, vf_{2x}], [\alpha f_{1x}, \alpha f_{2x}], [vf_{1v}, vf_{2v}]\}. \tag{26b}$$

We may now compute the generators of $\mathcal{W}$ modulo $\mathcal{J}$. First we note some facts concerning the higher order terms p_i. Recall that p_1 (p_2) is of order four (three) in x, y and α but that also y can only appear through $y^2 (= v)$. In addition the Taylor expansions cannot contain any terms purely involving powers of α (put another way $p_i(0, 0, \alpha) = 0$). To see this, just recall that the bifurcation equations are the projections onto e^{is} and e^{2is} but that these exponentials can only occur when accompanied by a coefficient x or y. Hence the lowest order terms in the expansion of

p_1 (p_2) are $v^2, \alpha^2 v, \alpha x v, x^2 v, x^4, x^3\alpha, x^2\alpha^2, x\alpha^3$ ($v^2, xv, \alpha v, x^3, x^2\alpha, x\alpha^2$). Bearing in mind these observations there results that (working mod $\mathcal{J}$)

$$[f_1, 0] = [\alpha x + \alpha v + \frac{16}{3} xv + 8x^3, 0] \qquad [vf_2, 0] = [\alpha v + 5v^2, 0]$$

$$[0, f_2] = [0, \alpha + 5v + \frac{31}{3} x^2] \qquad [xf_{1x}, xf_{2x}] = [\alpha x + \frac{16}{3} xv + 24x^3, \frac{62}{3} x^2]$$

$$[vf_{1x}, vf_{2x}] = [\alpha v + \frac{16}{3} v^2, 0] \qquad [\alpha f_{1x}, \alpha f_{2x}] = [\alpha^2 + \frac{16}{3} v\alpha, 0] \quad [vf_{1v}, vf_{2v}] = [\alpha v + \frac{16}{3} xv, 5v]. \tag{27}$$

From (23)-(27) and some manipulation with the generators in (27) there results that $RT(g + tp)$ is given by the right hand side of (19) which also shows that it is independent of t and hence that $RT(g + tp) = RT(g)$.

It remains to prove our claim. By inspection it is clear that

$$\mathcal{J} \subseteq \ [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] \tag{28}$$

and thus we must prove the reverse inclusion. By Nakayama's lemma, it is sufficient to prove that

$$[< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] \subseteq$$

$$\mathcal{J} + [< x^4 > \mathcal{M} + < v, \alpha > \mathcal{M}^3, < x^3 > \mathcal{M} + < v, \alpha > \mathcal{M}^2]. \tag{29}$$

If we look at the those parts of the generators (24) which arise from terms of the form tp_i, we can see that they all belong to the right hand-side of (29) and so that when verifying (29) it is permissible to take $tp_i = 0$. The proof now proceeds by observing that the module on the left side of (29) has the sixteen generators

$$[x^4, 0], \quad [v^3, 0], \quad [\alpha^3, 0], \quad [\alpha v^2, 0], \quad [xv^2, 0] \quad [\alpha x^2, 0], \quad [vx^2, 0],$$

$$[x\alpha^2, 0], \quad [v\alpha^2, 0], \quad [xv\alpha, 0], \quad [0, x^3], \quad [0, v^2], \quad [0, \alpha^2], \quad [0, xv] \quad [0, x\alpha], \quad [0, \alpha v]. \qquad (30)$$

We now expand the twenty-two generators of $\mathcal{J}$ in terms of the sixteen generators given in (30). Note that since we are taking $tp = 0$, we may replace f_i by g_i in (23)

This yields a 22×16 matrix. If we can show this has maximal rank 16 then the theorem will be proved. We do not actually present this matrix here, but it can easily be constructed. Taking the generators in the order given above, the first row, being the expansion of $x[g_1, 0]$, is (8,0,0,0,0,1,16/3,0,0,1,0,0,0,0,0,0); the next, the expansion of $v[g_1, 0]$, is (0,0,0,1,16/3,0,0,0,0,0,1,0,0,0,0,0,0) and so on. It is then not hard to show that this matrix has full rank, either by MATHEMATICA or by methods such as those used in [Golubitsky *et al* 1988, p190].

The usefulness of the preceding theorem is that it enables us to disregard the higher order terms in the bifurcation equations. We may henceforth confine ourselves to dealing in future with the reduced system

$$(g_1, yg_2) = (0, 0). \tag{31}$$

We may now proceed to our main result.

THEOREM 4.5 The codimension of (31) is seven.

More specifically, a universal unfolding of

$$\left(\alpha x + \alpha y^2 + 8x^3 + \frac{16}{3}xy^2, \alpha y + \frac{31}{3}x^2y + 5y^3\right) \tag{32}$$

is

$$\left(\alpha x + 4\beta x + \alpha y^2 - 2\beta y^2 - 6\gamma y^2 + 8x^3 + \frac{16}{3}xy^2 + \delta_1 + \delta_2 x^2 + \delta_3 x^3,\right.$$

$$\left.\alpha y + \beta y - 3\beta xy - 3\gamma xy + \frac{31}{3}x^2y + 5y^3 + \delta_4 xy + \delta_5 y^3\right), \tag{33}$$

where β, γ and $\delta_i (1 \leq i \leq 5)$ are the unfolding parameters.

PROOF Recall from §3 that the first step is to determine the tangent space $T(g)$, which is defined to be

$$T(g) = RT(g) \oplus \mathbb{R}\{(dg)Y_1, g_\alpha, \alpha g_\alpha, \alpha^2 g_\alpha \ldots\} \tag{34}$$

where $Y_1 = (1, 0)$ and thus $(dg)Y_1 = [\alpha + 24x^2 + \frac{16}{3}v, \frac{62}{3}x]$. Secondly, it is not hard to verify that the terms $\alpha^i g_\alpha, i \geq 2$ may be absorbed into $RT(g)$ and hence an easy calculation shows

$$T(g) = [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] + \mathbb{R}\{q_i, 1 \leq i \leq 7\} +$$

$$\mathbb{R}\{[\alpha + 24x^2 + \frac{16}{3}v, \frac{62}{3}x], [x+v, 1], [\alpha x + \alpha v, \alpha]\}. \tag{35}$$

However it is not hard to confirm that $3[\alpha x + \alpha v, \alpha] = 3q_1 + q_4 - q_7 + q_5 - (1/2)q_6$ and so

$$T(g) = [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] + \mathbb{R}\{q_i, 1 \le i \le 9\} \tag{36}$$

where $q_8 = [x+v, 1], q_9 = [3\alpha + 72x^2 + 16v, 62x]$. To determine a universal unfolding of g we see from Theorem 4.3 that the next step is to find a complementary subspace for $T(g)$ in $\overrightarrow{\mathcal{E}}(\mathbb{Z}_2)$. The idea is to write

$$\overrightarrow{\mathcal{E}}(\mathbb{Z}_2) = [< x^4 > + < v, \alpha > \mathcal{M}^2, < x^3 > + < v, \alpha > \mathcal{M}] +$$

$$\mathbb{R}\{[x^3, 0], [x^2, 0], [v^2, 0], [\alpha^2, 0], [xv, 0], [x\alpha, 0], [v\alpha, 0], [\alpha, 0], [v, 0], [x, 0], [1, 0],$$

$$[0, x^2], [0, \alpha], [0, v], [0, x], [0, 1]\}, \tag{37}$$

and then to observe that the third, fourth and seventh generators in the complementary subspace of (37) are in fact q_2, q_3 and q_1. We then proceed by expanding q_i $(4 \le i \le 9)$ in terms of the remaining generators in (37). This will lead to a 6×13 matrix. If we can augment this matrix by adjoining seven new rows in such a way that the new matrix is invertible, then we will have found the correct vectors to span a complementary subspace. The 13×13 matrix shown in Table 1. The first six rows are the expansions of the q_i while the last seven form the complementary subspace. Showing that this matrix is non-singular is straightforward and thus the theorem is proved.

REMARK The seven unfolding parameters have been labeled β, γ and $\delta_i(1 \leq i \leq 5)$. The parameter β corresponds to perturbations in the surface tension while γ correspond to perturbations in the phase speed condition. It interesting to speculate as to any possible physical significance of the other parameters.

5. Solution curves of the system

Now we have found our universal unfoldings we shall sketch the solution sets for a representative selection of the parameters. We briefly recall a couple of definitions.

DEFINITION 5.1 Consider an (α, x, y) coordinate system and let $\Gamma_\alpha = \{(\alpha, 0, 0) | \alpha \in \mathbb{R}\}$. Let

$$\mathcal{C} = \{(\alpha(t), x(t), y(t)) | t \in (-\delta, \delta)\} \subseteq \mathbb{R}^3$$

be a curve such that $\mathcal{C} \cap \Gamma_\alpha = \{(\alpha_o, 0, 0)\}$; then $(\alpha_o, 0, 0)$ is said to be a *primary bifurcation point* and $\mathcal{C}$ is said to be a *primary bifurcation curve.*

If

$$\mathcal{D} = \{(\overline{\alpha}(t), \overline{x}(t), \overline{y}(t)) | t \in (-\delta, \delta)\} \subseteq \mathbb{R}^3$$

	[*,0] where * is								[0,*] where * is				
	x^3	x^2	xv	$x\alpha$	α	v	x	1	x^2	α	v	x	1
q_4	24		16	3									
q_5									31	3	15		
q_6	48								62				
q_7			16								15		
q_8						1	1						1
q_9		72			3	16						62	
						-2	4					-3	1
						-6						-3	
								1					
		1											
	1												
												1	
											1		

Table 1

is another continuous curve such that $\mathcal{D} \cap \mathcal{C} = \{(\overline{\alpha}, \overline{x}, \overline{y})\} \not\subset \Gamma_\alpha$, then $(\overline{\alpha}, \overline{x}, \overline{y})$ is called a *secondary bifurcation point* and $\mathcal{D}$ is called a *secondary bifurcation curve.*

We shall draw the bifurcation diagrams for a representative selection of the unfolding parameters. Recall from §2 that when β and γ are both zero, the null space of the linearized equation is two dimensional and the situation is that of bifurcation from a double eigenvalue covered in, for example [Dancer 1973, Rabinowitz 1971]. If only the parameter γ is perturbed from zero the null space remains two dimensional. It is only when β is perturbed that the double eigenspace splits and the case is covered by the classical bifurcation from a simple eigenvalue [Crandall & Rabinowitz 1971]. We shall concentrate on the most physically relevant case, when the only unfolding parameters present are β and γ. The equations become

$$\alpha x + 4\beta x + \alpha y^2 - 2\beta y^2 - 6\gamma y^2 + 8x^3 + \frac{16}{3} x y^2 = 0, \tag{38a}$$

$$\alpha y + \beta y - 3\beta x y - 3\gamma x y + \frac{31}{3} x^2 y + 5y^3 = 0. \tag{38b}$$

Obviously one solution curve is that along which $y = 0$, that is

$\alpha = -4\beta - 8x^2$, which represents a primary bifurcation curve which branches from Γ_α at $\alpha = -4\beta$. We shall denote this by $\mathcal{C}_2$, solutions lying on this have period π. When we come to consider other solutions of the system, first cancel y from (38b) and then eliminate α between the equations. This yields the equation

$$3\beta x + 3(\beta+\gamma)x^2 - 3(\beta+2\gamma)y^2 + 3(\beta+\gamma)xy^2 - \frac{7}{3}x^3 + \frac{1}{3}xy^2 - 5y^4 - \frac{31}{3}x^2y^2 = 0. \qquad (39)$$

We now proceed to sketch the solution curves of (39) and the corresponding ones of (38) for various representative values of the parameters. Note that a solution curve of (39) will always pass through the origin and that this corresponds to a primary bifurcation point at $\alpha = -\beta$. In the event that (39) possesses a curve which intersects the x-axis, this corresponds to a secondary bifurcation point on $\mathcal{C}_2$ at which a doubling of the minimal period of the interface, from π to 2π occurs.

Let us first look at the completely unperturbed case ie when β and γ are both zero. Then the solutions to (39) are depicted in Fig 2(a) and those to (38) in Fig 2(b). We see there are two primary curves both bifurcating from the origin.

Next we may consider what happens when $\beta = 1$ but $\gamma = 0$. The solution curves to (39) and (38) are depicted in Fig 3. It is then clear that in this case the primary curve $\mathcal{C}_1$ is a closed loop which also meets $\mathcal{C}_2$ at a secondary bifurcation point. In addition there is another unbounded secondary curve.

Now let us set $\beta = -1$ but retain $\gamma = 0$. We then can see from Fig 4 that $\mathcal{C}_1$is just an unbounded

β=0, γ=0

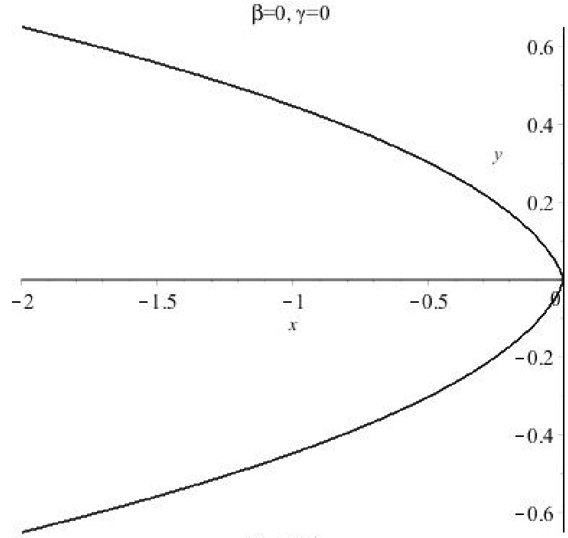


Fig. 2(a)

$\beta = 0, \gamma = 0$

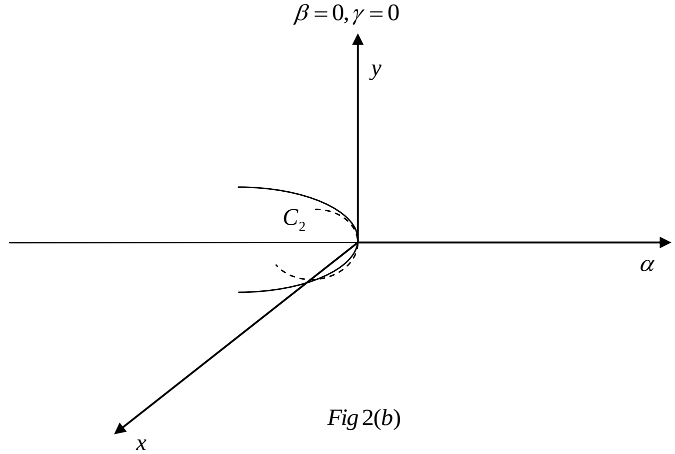


Fig 2(*b*)

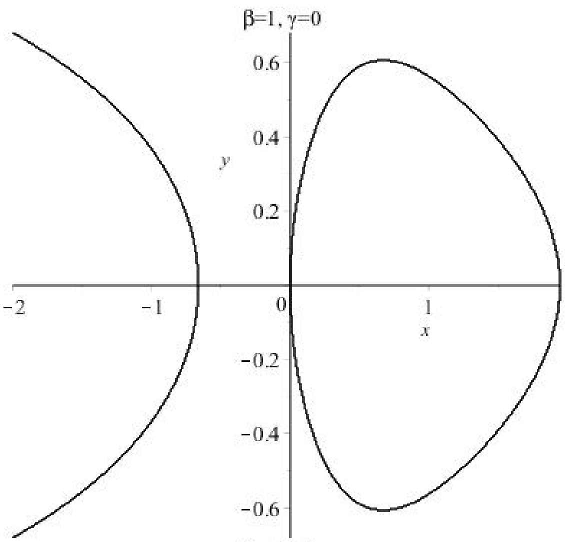


Fig. 3(a)

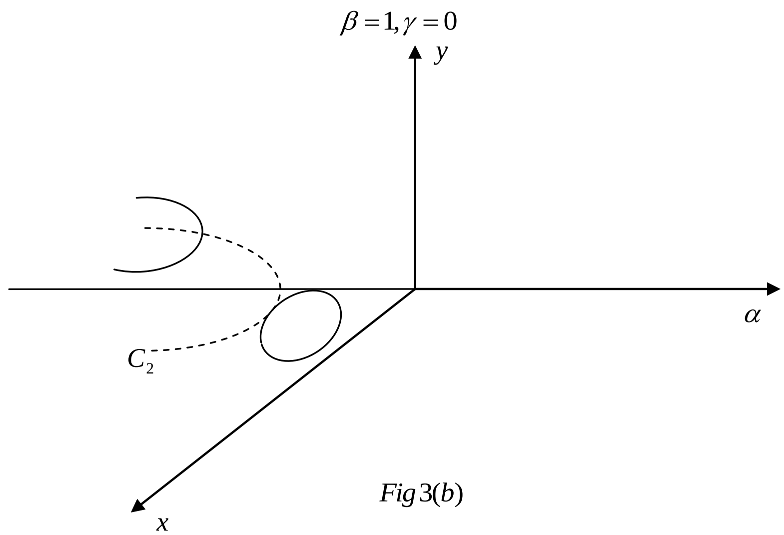


Fig 3(*b*)

curve and secondary bifurcation does not take place.

For our next set of results, set $\beta = 0$ and $\gamma = 1$ so that γ but not β is perturbed. The solution curve are depicted in Fig 5 and we see that in addition to $\mathcal{C}_2$ there is a loop and also an unbounded bifurcation curve. Note that in this case all bifurcations are from the origin. Now set $\beta = 0$ and $\gamma = -1$. Then we see from Fig 6 that $\mathcal{C}_1$ consists of a figure of eight and in addition there is an unbounded secondary curve. Next let us set $\beta = 1$ and $\gamma = -1$ so that both parameters are perturbed. The solutions are configurated in Fig 7. Finally, let us very briefly examine the solution portraits of the equations for the other unfolding parameters. If we set the fully unfolded system (33) to zero, then we see that either $y = 0$ in which case the solution curve is

$$\alpha x + 4\beta x + 8x^3 + \delta_1 + \delta_2 x^2 + \delta_3 x^3 = 0. \tag{40}$$

Observe that it is not a bifurcation curve unless $\delta_1 = 0$. Otherwise, the augmented form of (39) is

$$\begin{aligned} 3\beta x + 3(\beta+\gamma)x^2 - 3(\beta+2\gamma)y^2 + 3(\beta+\gamma)xy^2 - \frac{7}{3}x^3 + \\ \frac{1}{3}xy^2 - 5y^4 - \frac{31}{3}x^2y^2 + \delta_1 + \delta_2 x^2 + \delta_3 x^3 - \delta_4(x^2+xy^2) - \delta_5(y^4+xy^2) = 0. \end{aligned} \tag{41}$$

Let us first set $\beta = \gamma = \delta_1 = 1$ and $\delta_4 = \delta_5 = -2$. Then the curve (40), corresponding to $\mathcal{C}_2$,

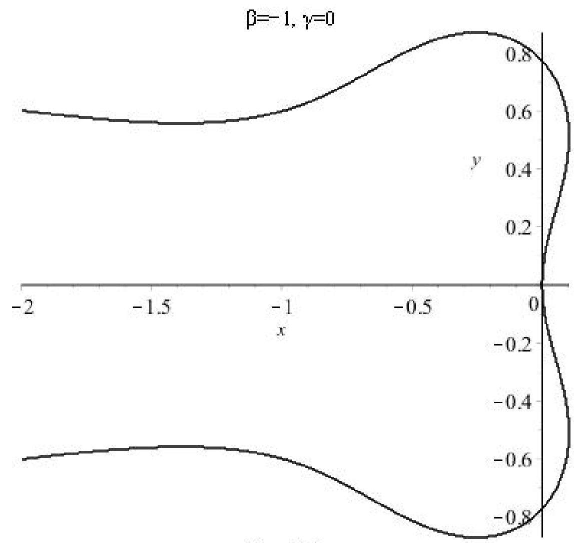
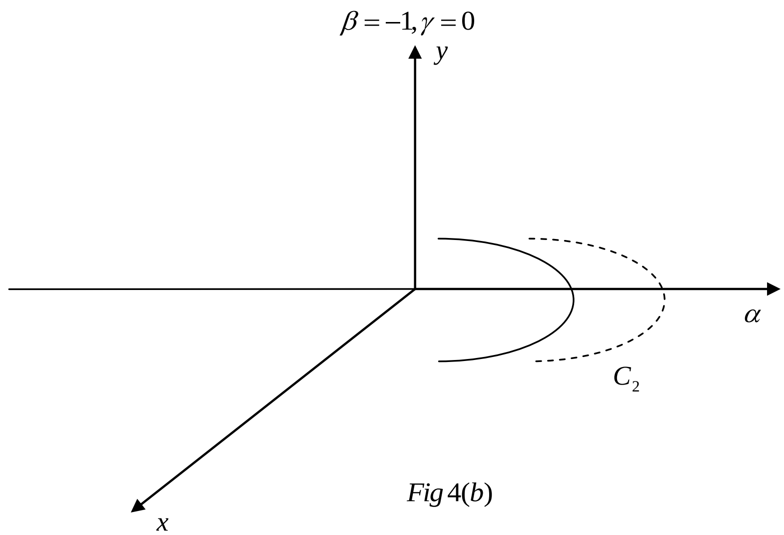


Fig. 4(a)



$Fig\,4(b)$

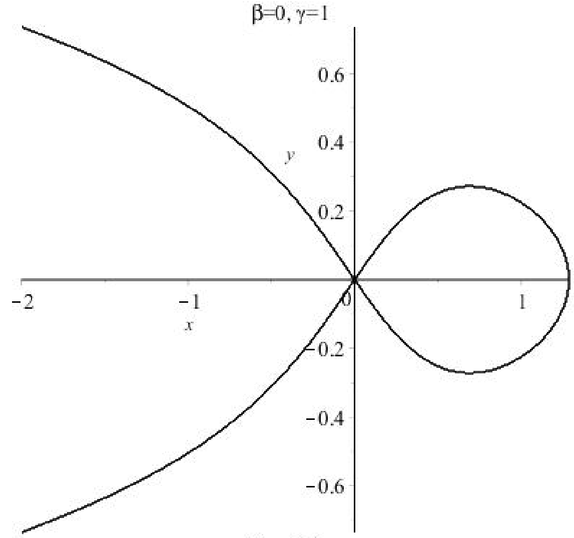


Fig. 5(a)

$\beta = 0, \gamma = 1$

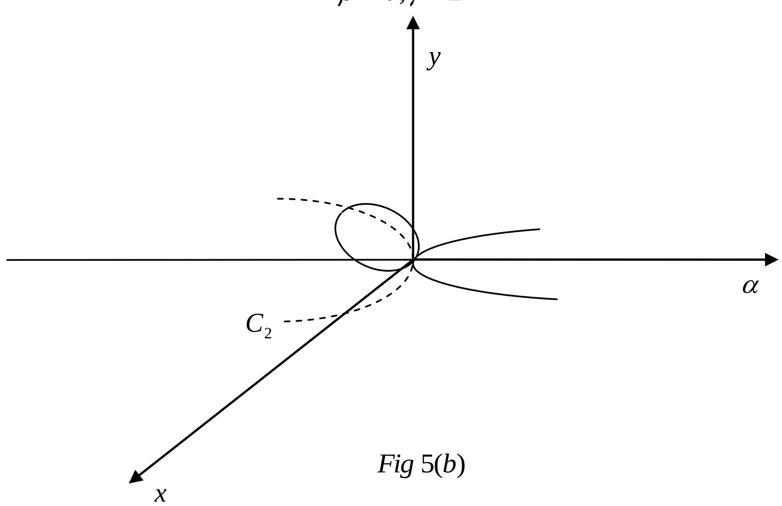


Fig 5(*b*)

β=0, γ=−1

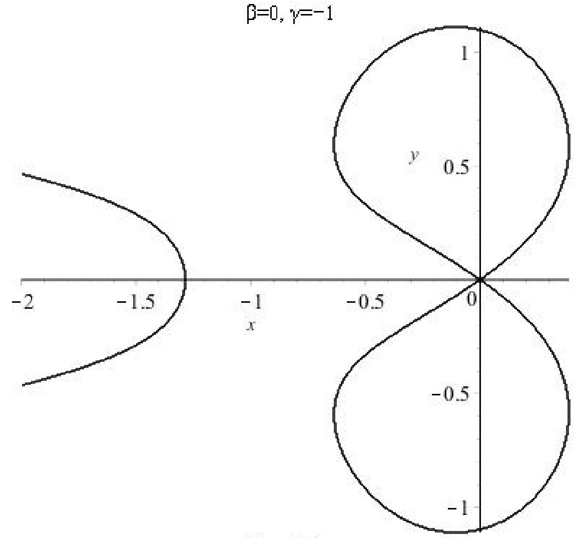
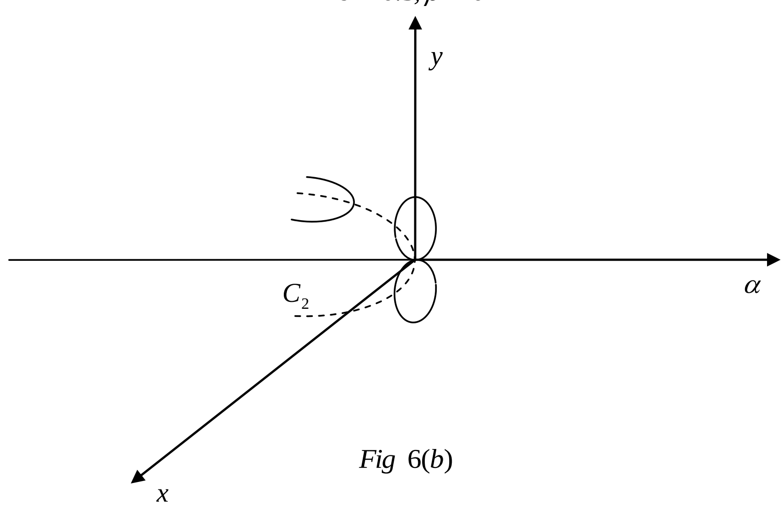


Fig. 6(a)

$b = 0.3, \beta = 0$



$Fig\ \ 6(b)$

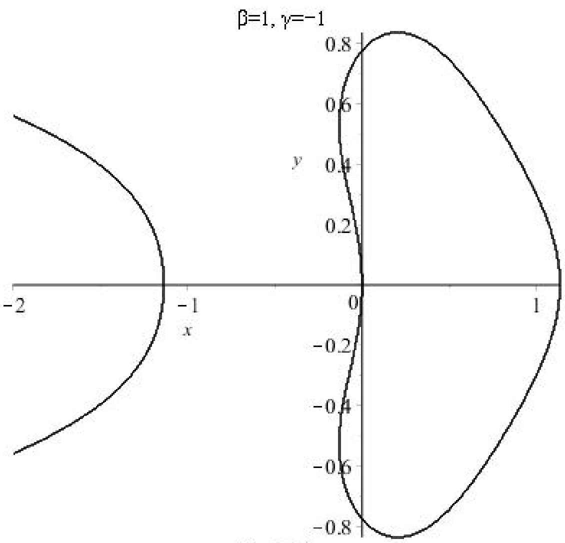
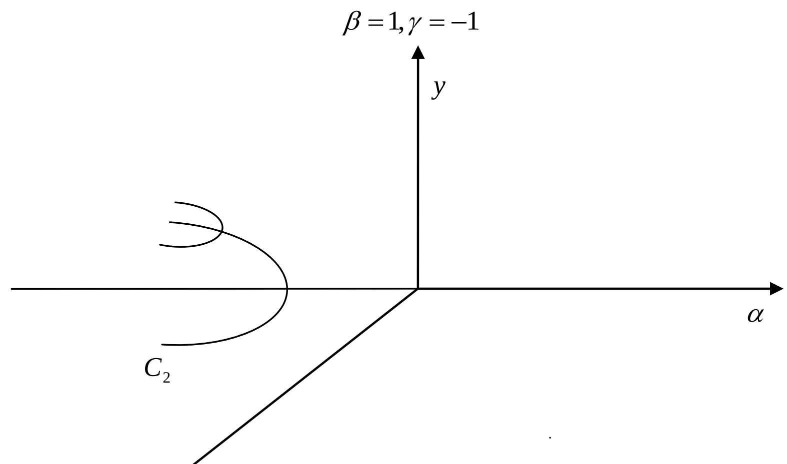


Fig. 7(a)



$Fig\ 7(b)$

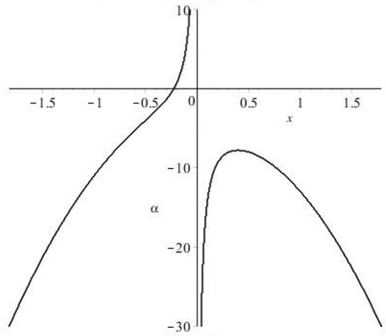


Fig. 8(a)

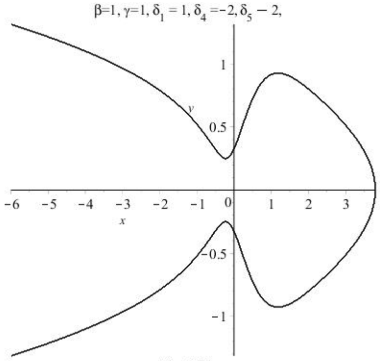


Fig. 8(b)

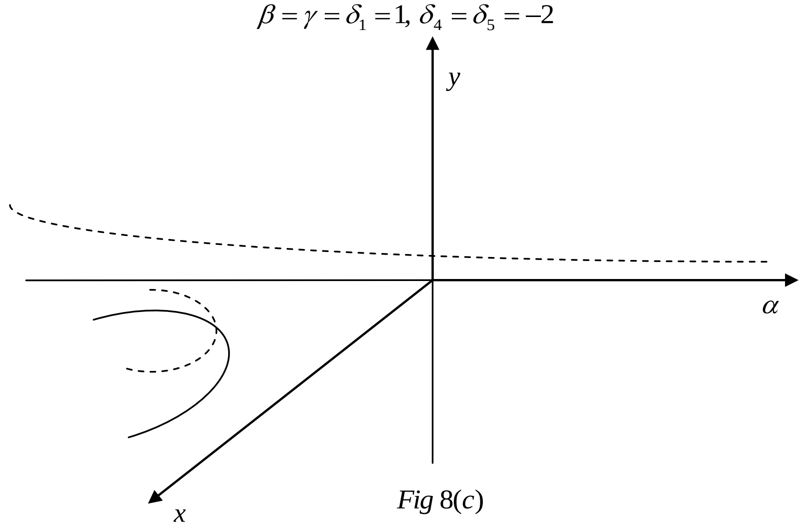


Fig 8(*c*)

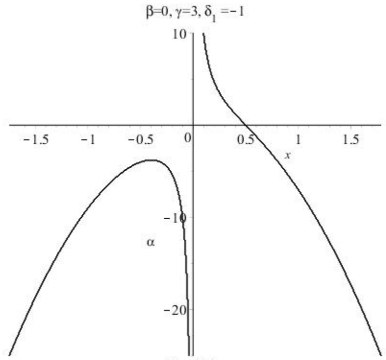


Fig. 9(a)

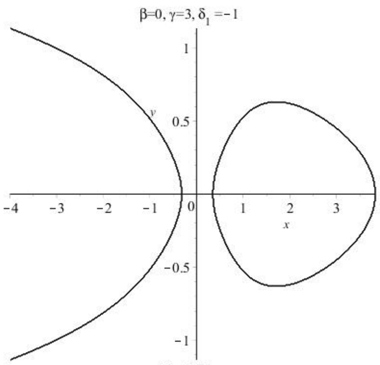
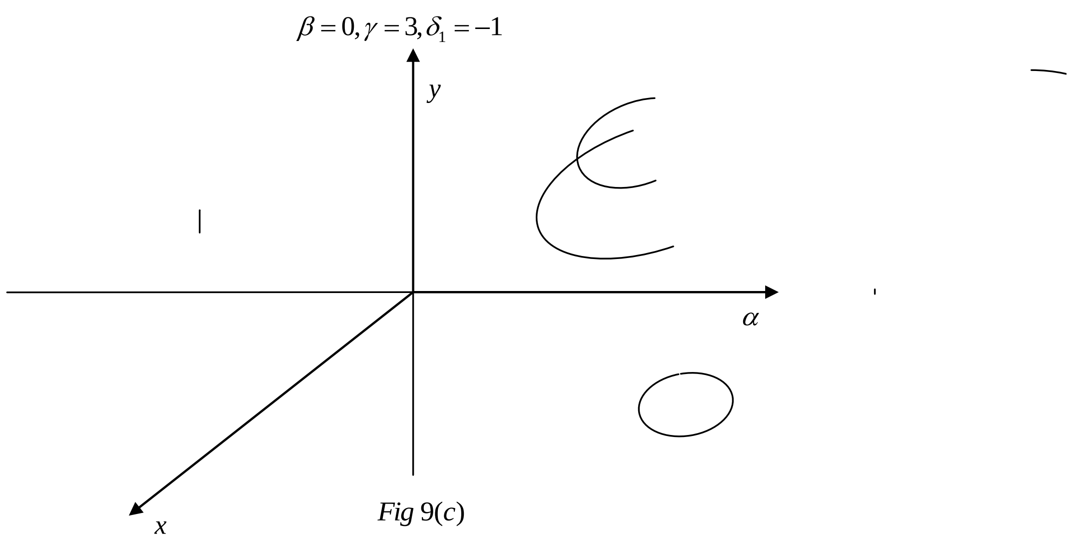


Fig. 9(b)



$Fig\ 9(c)$

is $\alpha x + 4x + 8x^3 + 1 = 0$. This is depicted (in the α-x plane) in Fig 8(a). Observe that x can never be zero, but that it can assume all other non-zero values. Thus this not a bifurcation curve but since x can take values arbitrarily close to zero, it could be described as a "bifurcation from infinity". The curve (41) is depicted in Fig 8(b) (in the x-y plane). The solutions to the unfolded equations are then as in Fig 8(c). Finally set $\beta = 0, \gamma = 3, \delta_1 = -1$. Then the curve (40), is depicted in Fig 9(a) while the curve (41) is depicted in Fig 9(b). The solutions to the unfolded equations are then as in Fig 9(c). Observe that in this case there are two "secondary bifurcation"curves, one bounded and one unbounded.

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