Kingston University

Direct Numerical Simulation of Turbulent Flows over Complex Geometries

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Direct Numerical Simulation of Turbulent Flows over Complex Geometries

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ABSTRACT

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by Jony Castagna

The aim of this work is to extend an existing CFD solver, named Shock/Boundary-Layer Interaction (SBLI) code, to include a fully 3D curvilinear capability in order to perform direct numerical simulation (DNS) of turbulent flows over complex geometries. The SBLI code solves the compressible Navier-Stokes equations by the finite difference method and uses the body-fitted curvilinear coordinate system approach to treat complex geometries. The extended version of the code has been used to perform a DNS of a channel flow with longitudinally ridged walls and a DNS of a turbulent flow over an axisymmetric hill geometry. Validation and comparison with previous experimental data and numerical results are also presented.

In the first part of the work, the Navier-Stokes equations are presented in a strong conservation form and test validations of the code extension have been carried out such as free stream flow preservation on a wavy grid and a laminar plane channel flow on a skewed mesh. The free stream preservation test consists of a uniform flow computation on a cosinusoidal mesh and the objective is to evaluate the velocity components changes from their initial values due to the effect of a highly skewed mesh. The maximum discrepancy found is around 10^{-16} . For the laminar plane channel flow simulation on a skewed mesh, the purpose is to verify the symmetrical propriety of numerical errors obtained in the velocity components while the main flow direction and the position of the walls are altered in rotation around the three physical coordinates. The symmetry of the numerical error is found to be well preserved as expected.

The second part of the work contains DNS of laminar and turbulent flows in a channel with longitudinally ridged walls at different Reynolds numbers. The goal is to investigate the effect of ridged walls on the turbulent flow behavior and to provide quality DNS data for assessing other numerical simulations, such as Large Eddy Simulation (LES) and Reynolds-Averaged Navier Stokes (RANS) modeling. Two Reynolds numbers have been simulated ($Re_{\tau} = 150$ and $Re_{\tau} = 360$, based on a reference velocity $u_{\tau} = \sqrt{\delta/\rho_b(-dP/dx)}$, the bulk density and the wall viscosity) on a domain of $1.25\pi\delta \times 2\delta \times 0.375\pi\delta$ in the streamwise, wall normal and spanwise directions, respectively. This domain is similar to the minimal flow unit for a turbulent plane channel flow. Comparisons with previous experimental data and numerical prediction have show good agreement for the $Re_{\tau} = 150$ case and a similar flow dynamics for the $Re_{\tau} = 360$ case. In general, the effects of ridged walls on the turbulent flow, like the reduction of the normal Reynolds stress peak values, seems to be smaller when the Reynolds number increases.

The third part of this work describes the main simulation of this thesis. DNS of a turbulent flow around an axisymmetric hill is carried out in order to investigate the three-dimensional boundary-layer flow separation which occurs behind the hill. Different domain sizes and grid resolutions have been tested up to a maximum of about 54 million points. A methodology for generating inflow conditions has been implemented and tested. Results are compared with previous experimental and numerical studies. Due to a low Reynolds number used ($Re_{\delta^*} = 500$, only 5% of an experimental simulation), the time averaged separation bubbles is much bigger and the flow seems to have a laminarisation process due to a strong adverse pressure gradient presented. A small recirculation bubble detected on the top of the hill seems to be the cause of the earlier separation of the turbulent boundary layer and, then, the bigger separation observed. However, similar to the full Reynolds number experiment, same flow

dynamics, consisting in the formation of a counter rotating vortex pair merging in the streamwise current, have been captured well.

The final part of the work presents an extension of the single-block SBLI code to a multiblock version. A pre-processor program has been developed in order to simplify the treatment of the interface between different blocks and a description of the algorithm is also given. As a demonstration study, DNS of a square jet in a turbulent cross flow has been performed at two Reynolds numbers ($Re_{\delta^*} = 1000$ and $Re_{\delta^*} = 2000$) and different jet to cross flow velocity ratios. Compared with the available data, the results are in good agree, despite the lower Reynolds number used (half of value simulated in the available data).

In conclusion, a fully 3D version of the SBLI code has been successfully derived and tested for various flow configurations. The 3D curvilinear capability has also been implemented and tested by simple, but not trivial, test cases. An option for simplified treatment of Cartesian mesh has been implemented and tests have shown a factor of 2 speedup in overall performance. Two main simulations have been carried out and for the turbulent flow in a ridged channel, the results are in good agreement with published data, while, for the flow over an axisymmetric hill case, simulation is compared qualitatively well and the noticeable discrepancies are primarily due to a reduced Reynolds number conditions. The code has also been successfully extended to a multiblock version and demonstrated on a two-block domain for a jet in cross flow case. Future works includes simulations of the hill problem at higher Reynolds number and LES extension of the SBLI code to fully 3D curvilinear capability.

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> "Lontani da qui, ma stretti nel cuore. Grazie Mamma e Papà".

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Nomenclature

Roman Symbols

A	Amplitude of sinusoidal wave stretching
C_p	Pressure coefficient, constant-pressure specific heat
С	Speed of sound
E	Energy
E_{tot}	Total energy
e	Specific energy
Н	Half channel height, hill height
Ι	Integral length scale, sum of minor determinant of metric matrix
J	Jacobian of metric matrix
Jac	Minor determinant of metric matrix
Jacodet	Metric terms multiplied for the inverse of the Jacobian
k	Turbulent kinetic energy, thermal conductivity, von Kármán constant,
	turbulence modes
L	Domain dimensions
l	Integral length scale
M	Mach number
Ν	Number of points
Р	Pressure

P,Q,D	Carpenter matrixes
Pr	Prandtl number
Procs	Number of processors
q_i	Thermal flux
R	Constant of ideal gas, two point correlation
	Respectively
Re	Reynolds number
res	Metrics terms
Т	Temperature, computational time
t	Physical time
U	Velocity vector
u, v, w	Streamwise, normal and spanwise instantaneous velocity, respectively
w	Pulsation of sinusoidal wave stretching
x, y, z	Streamwise, normal and spanwise directions, respectively
$\overline{x},\overline{y},\overline{z}$	Streamwise, normal and spanwise directions, respectively, for
	uniform meshing

Greek Symbols

β	Entropy splitting factor
$\overline{\Delta x}, \overline{\Delta y}, \overline{\Delta z}$	Grid spacing in x, y, z -direction, respectively, for uniform meshing
Δt	Time step
δ	Reference length
δ_r^*	Dimensional reference length
ε	Average rate of dissipation
γ	Heat capacity ratio

λ_2	Second largest eigenvalue of the of $S^2 + \Omega^2$
η	Kolmogorov scale
μ	Fluid viscosity
μ_r^*	Dimensional reference fluid viscosity
ν	Kinematic viscosity
π	Pi Greek number
ρ	Fluid density
$ ho_r^*$	Dimensional reference fluid density
$ au_{ij}$	Stress tensor
$\widetilde{\omega}_x$	Streamwise average vorticity (Favre average)
ξ,η,ζ	Computational coordinates
ξ,η,ζ	Minor determinant of metric matrix

Other Symbols

1	The fluctuating part
_	Reynolds average
~	Favre average
$\langle \rangle$	Average in spanwise direction
*	Dimensional variable

Subscripts

1, 2, 3	Indices (streamwise, wall-normal and spanwise directions, respectively)
cl	Centerline value

i, j, k	Indices (streamwise, wall-normal and spanwise directions, respectively)
max	Maximum value
r	Reference value
w	Wall value
∞	Freestream value
tot	Total
rms	Root mean square
x,y,z	Streamwise, wall-normal and spanwise direction, respectively

Superscripts

-	Reynolds-averaged
~	Favre-averaged fluctuation
*	Dimensional quantities
+	Quantity in wall units

Abbreviations

AUs	Allocation Units
CFD	Computational Fluid Dynamic
CFL	Courant-Friedrichs-Lewy
CRVP	Counter Rotating Vortex Pair
DNS	Direct Numerical Simulation
HPC	High Performance Computing
LES	Large Eddy Simulation

LDV	Laser Doppler Velocimeter
MPI	Message Passing Intervface
N- S	Navier-Stokes
RANS	Reynolds Averaged Navier Stokes
RMS	Root mean square
r.h.s.	right hand side
SBLI	Shock Boundary Layer Interactions
TKE	Turbulent Kinetic Energy

Chapter 1

Introduction

1.1 Background and Motivation

1.1.1 Computational Fluid Dynamics

Over the last few decades, Computational Fluid Dynamics (CFD) has become a powerful tool for investigation of complex physical phenomena. Ranging from fundamental research to practical applications, CFD is successfully growing as an alternative solution tool to the traditional theoretical and experimental approaches and, considering the increasing computer technology, great expectations are foreseen for the near future. For example, the advance of parallel computing has given the opportunity to explore very complex phenomena, like turbulence, in the smallest details. Moreover, it is possible to simulate extreme flow conditions generally difficult to achieve in a normal wind tunnel, for example: it is possible to reproduce the behavior of very hot fluid, like plasma, or very high speed flows, typical of hypersonic flight conditions (see Aluri *et al.*, 2008). The increasing popularity of this new discipline of the science is reflected in the increasing number of investigators and CFD codes developed around the world. The CFD approach consists in solving the governing equations of fluid dynamics by numerical simulations. This requires a discretization of the above equations in space and in time, followed by an integration on a well defined computational domain. However, the computational effort required to correctly solve the equations can be very expensive. Some simplifications, based on different physical approaches, are usually adopted.

When the level of detail required for the solution of the fluid dynamics problem is not very important, that is we are mainly interested on the average values of the flow field quantities (density, velocity, etc.), a common approach is the use of the RANS equations. These equations are obtained by decomposing the flow quantities into a time averaged (\bar{u}) and a fluctuating component (u'), e.g. $u = \bar{u} + u'$, and by time averaging the resulting equations. This approach reduce greatly the grid requirements, but its main drawback is the loss of accuracy of the solution. Moreover, the closure of the RANS equations requires the use of turbulent models which correlate the Reynolds turbulent stresses to the velocity field. These models are often valid only for specific flow problems and for a small range of Reynolds numbers.

A more accurate approach, which has become more popular in the last years, is represented by the Large Eddy Simulation technique. Here the Navier-Stokes equations are filtered using a mathematical filter and the introduction of a *subgrid model*. The level of detail of the simulation is higher than that in the RANS approach, but also the computational effort is greater. The subgrid models are often valid for a wide range of Reynolds numbers, but they can require a very high grid resolution for some particular kind of flow, like in presence of boundary layer separation.

When a very high level of detail is required, the DNS is the ideal approach, since the conservation equations are solved in the exact form without any approximation or modeling. As presented in the next sections, the computational costs associated with the DNS for a typical of real applications are still prohibitive today. However, DNS can be used as an ideal tool for the investigation of fluid dynamics problems at low Reynolds numbers.

Whatever approach is used, the methods to discretize the fluid dynamics equations on the computational domain are several: finite element, finite volume, finite difference and spectral methods. Each of them has advantages and disadvantages, but finite difference and spectral methods are the most commonly used for direct numerical simulations of turbulent flows, since they allow an easy implementation of high-order, and hence low dissipative, schemes. Moreover, special attention needs to be paid to the particular fluid dynamics conditions, for example: in case of incompressible flows a common issue is the decoupling of the pressure field from the velocity field, solved by the use of a staggered grid (see Patankar, 1980 and Versteeg and Malalasekera, 1995). On the other hand, in case of high compressible flows, where shock waves can occur, shock capturing schemes are necessary in order to predict the position and strength of the shock discontinuity with enough accuracy, meanwhile retaining a smooth solution in the rest of the flow field. Many numerical schemes have been developed in several ways, increasing the CFD knowledge and expanding its application to a large number of different fields (multiphase flow, combustion, etc.). A better introduction to CFD can be found in Anderson (1995) and Ferziger and Perić, (2002).

1.1.2 Turbulence Challenges

Turbulence is an intrinsically three-dimensional phenomenon, unsteady and irregular in space and time and it is governed by the same mass, momentum and energy conservation equations of the fluid flow. It plays a fundamental role in the understanding of several effects like drag reduction, lift, boundary layer separation, etc.

Due to its intrinsic difficulty, a complete theory of the turbulence does not still exist, although there has been a large number of researchers dedicated to this topic in the last century. The analytical solution of the Navier-Stokes equations in case of a turbulent flow is possible only after some simplifications, like time average of the physical variables, self similarity hypothesis, etc. (see Schlichting, 2000). Some theories, as Kolmogorov's theory on isotropic turbulence (see Mathieu, 2000), have helped our understanding in some aspects of the phenomenology. For example, from Kolmogorov's theory it is possible to estimate the magnitude order of the largest and smallest scale for a given Reynolds number. A lack of knowledge in wall-bounded flows is still present and semi-empirical approaches, such as the *law of the wall* and the *log law*, which have been the base for approximation models, can easily fail in presence of separated flows or secondary flows.

Turbulence has well defined statistical proprieties. For example, in the turbulent boundary layer over a flat plate experimental results prove that the statistical proprieties are the same for all the experiments and some empirical statistical turbulent models have been proposed to correctly reproduce the experimental data. Unfortunately, if a separation occurs, like over a wing surface, the statistical results observed are completely different from the flat plate and new models need to be implemented. Today's turbulence challenge is to develop a general model which is valid for all kind of flow and for all Reynolds number range.

1.1.3 DNS Challenges

Simulation of turbulent flows is a very difficult task, especially due to the wide range of energy frequencies presented at a given Reynolds number. Figure 1.1 shows a typical graph of the energy spectrum (energy E versus modes k) of an isotropic turbulent flow: the energy is transferred from the largest structures (low frequencies) to the smallest eddies (high frequencies) where the kinetic energy is dissipated by viscous effects. To obtain the correct energy decay, it is essential to reproduce all the scales. From Kolmogorov's theory (see Mathieu, 2000), an estimation of the smallest eddy length presented in isotropic turbulence, called the Kolmogorov scale, is $\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$, where ν is the kinematic viscosity of the fluid and ϵ is the average rate of dissipation per unit mass. From the same theory, ϵ is estimated to be $\epsilon \approx U_{rms}{}^3/l$, where U_{rms} is the root mean square of the velocity and l the largest scale, called the integral scale. From ν and l it is possible to estimate that the total number of computational nodes necessary to correctly solve all the frequencies in the flow is $N \geq Re^{9/4}$. For practical applications with a $Re \approx 10^6$ the number of points required is $N \geq 31 \times 10^{12}$, which is much beyond of today's computer capability and, considering the Moore's law on the trend of computing capability, the DNS will became a common practice only after several decades from today. However, for fundamental research at low Renumbers, the DNS approach is ideal, considering the high quality of well resolved simulations, often even better than experimental results. Moreover, a big advantage is the possibility to analyze all the flow properties in each point of the domain and at each time step of the temporal integration.

As cited above, in DNS, contrarily to RANS and LES techniques, the original form of the conservation equations is preserved. However, special attention needs to be paid to the numerical schemes implemented. In fact, DNS requires very high resolution even in the time coordinate and then it requires small time step in order to guarantee an accurate solution. On the other hand, a long integration time can significantly increase the inevitable numerical error due to the approximation of the discretized equations. An unbounded growth of the error can compromise stability of the solution. Many techniques have been employed in the last few years in order to control the instability due to the numerical errors. A promising method is the so called entropy splitting technique (Yee *et al.*, 1999), based on the splitting of the conservation equation by an entropy function. See Section 3.3 for further details.

DNS requires the use of high-order schemes and stable algorithms, but high-order schemes are difficult to adapt to complex geometries. The challenge is to find a compromise between those two requirements in order to reduce the computational costs.



Figure 1.1: Energy spectra for an isotropic turbulent flow.

1.1.4 Motivation

In the last few decades, high-order finite difference methods have become the preferred technique for the direct numerical simulation of turbulent flows. In fact, spectral methods, despite the high-order achievable, are difficult to adapt to complex geometry, while the finite volume method is difficult to implement for orders higher than second. However, most of the real applications occur in very complex geometries and the use of finite difference methods to these systems is not trivial.

In this work, we have extended an existing code for quasi-3D curvilinear geometry to fully 3D. The SBLI code was originally developed for curvilinear variation in the xy plane only, with a linear stretching feature in the z-direction. Since many practical problems involve curvilinear meshes in all the three spatial directions, like for the flow over a real wing or the separation behind a three-dimensional obstacle, the necessity of a fully 3D curvilinear version of the code is evident. However, this upgrading requires a substantial change of the main subroutines of the SBLI code and validation test cases are necessary. Laminar channel flows and free stream preservation on highly distorted meshes have been considered as preliminary tests, but better validation can be achieved by performing turbulent flow simulations.

Two main simulations have been carried out: a) a turbulent flow in a channel with longitudinally ridged walls (hereafter named as *ridged channel*); b) a turbulent flow over an axisymmetric hill.

In the first case, the geometry is still not fully curvilinear as the grid is only stretched in the z-y plane. However, the results of the simulation can be compared with the results obtained from the previous version of the code in order to verify the preservation of the solution. Moreover, we are interested in the effect of the ridged walls on the turbulent flow at different Reynolds numbers. Note that a large recirculation is present over the ridge, which is not captured by simple RANS turbulent models like the k- ϵ .

In the second case, the geometry includes spatial variation in all the three coordinates. This represent a good test for the fully 3D version as it involves many of the components of the metric relationship between curvilinear and orthogonal mesh. The main physical interest is in the understanding of the three-dimensional boundary layer separation mechanism which occurs behind the hill and the main challenge is the localization of the region where the separation occurs. In fact, this region seems to lie in a thin layer on the leeside of the hill and very high resolved simulations are required. A DNS can provide not only a clear understanding of the mechanism involved, but also numerical results which can be used for validations of new turbulent models for RANS and LES techniques.

Finally, considering that most of the practical engineering applications involves very complex geometries, a multiblock version of the code seems essential for future users of the code. Contrary to finite volume based solvers, where unstructured meshes can be used, finite difference based solvers require, in most of the cases, the use of structured meshes. For complex geometries it is necessary to use multiblock structured meshes which require the development of a special grid generation software. This is still an active field of today's CFD research.

1.2 Literature Review

In this section we give a brief review of the literature regarding the main topics of the present work: the 3D formulation of the body-fitted coordinate method and the 3D boundary layer separation. An extended review on the turbulent flow in a ridged channel and on the turbulent flow over an axisymmetric hill, which represent the major applications of this work, can be found in the introduction section of Chapters 5 and 6, respectively. A special section is dedicated to the SBLI code with a brief description of its main features and previous successful applications.

1.2.1 3D Curvilinear Formulation

The body-fitted coordinate method is probably the oldest method to solve the Navier-Stokes equations in complex geometries (Anderson 1995). Extensive introduction on this topic can be found in any text book of computational fluid dynamics (see Anderson, 1995 and Ferziger and Perić, 2002. See also Chapter 2 of this thesis), while a brief introduction is given here. The idea consists in the transformation from a curvilinear generic physical domain (x, y, z) to an orthogonal regular computational domain (ξ, η, ζ) , where the finite difference schemes can be easily applied. However, as presented by Pulliam and Steger (1978), while the above procedure works well for a 2D transformation, some problems occur in the 3D case if a finite difference central scheme is applied. In fact, even the preservation of a simple free stream in a wavy grid can be compromised (see Section 4.1). For moving meshes the Geometric Conservation Law (GCL) has to be verified, while for static mesh some metric terms (i.e. $\partial x/\partial \xi$, $\partial y/\partial \xi$, etc.) need to cancel each other. Pulliam and Steger (1978) suggested a simple averaging procedure to preserve the free stream. However, while this approach works well for a second-order scheme, the extension to high-order formulation is not trivial. Thomas and Lombard (1979) suggested a simpler approach based on a conservative reformulation of the previous metric relationship, which can be easily applied to high-order scheme as presented in Visbal and Gaitonde (2001, 2002). In this work we have used a slightly different formulation based on the conservative hyperbolic form of the Navier-Stokes equations. The free stream is well preserved and manipulations of the metric terms are not necessary.

1.2.2 3D Turbulent Boundary Layer Separation

Turbulent boundary layer separation occurs in many practical engineering applications like around a car profile, airplane wing surface, turbine blade junctions, etc. The separation affects the drag around those objects and then the performance in terms of speed of the car, airplane or turbine, and in term of the engine efficiency. To understand why the separation occurs and how to control it is essential in order to reduce the drag and gain on performance. The flow around a bluff body like a cube or a hill shape, occurs with 3D flow separation and it can help to understand the basic mechanism of the turbulent flow separation.

A good introduction on the boundary layer theory and boundary layer separation can be found in any text book of turbulence (see Schlichting 2000 and White 1991 for example). Commonly, the term "separation" is intended as the entire process of departure, breakaway or breakdown of the turbulent boundary layer (Simpson 1996), which normally occurs in presence of strong adverse pressure gradients. However, while for a bluff body the separation point is normally well defined (see Castro and Robins, 1977 and Coceal *et al.*, 2006), in the presence of complex aerodynamic shapes, like airplane wings or turbine blades, the separation point is often not even fix in time and its identification represents a very hard task for today's aerodynamicists. Simpson proposed a set of quantitative definitions for the separation point (Simpson 1996) based on the fraction of time that the flow moves downstream (γ_{pu}). Four points are identified (see Figure 1.2): *incipient detachment* (ID), where the backflow occurs 1% of the time ($\gamma_{pu} = 0.99$); *intermittent transitory detachment* (ITD) where the backflow occurs the 20% of the time ($\gamma_{pu} = 0.80$); *transitory detachment* (TD) where the backflow occurs the 50% of the time ($\gamma_{pu} = 0.50$) and *detachment* (D)



Figure 1.2: A flow model for the boundary layer separation criteria.

criteria above described are not always easy to identify. When the separation occurs in a very thin layer, like behind a hill obstacle (see Simpson *et al.* 2002, Byun *et al.* 2004 and Byun and Simpson 2006), very sophisticated equipments are required in order to capture the correct flow field and topology.

As cited before, the dependency of the mean streamwise velocity from the wall normal distance, for a fully developed turbulent flow on a flat plate, is described by the law of the wall and the log law $(u^+ = y^+ \text{ and } u^+ = \log(y^+)/k + b)$, respectively, where u^+ and y^+ are the dimensionless mean velocity and wall normal distance, k = 0.41 is the von Karman constant and b is an empirical constant normally found to be ≈ 5.5).

The effect of the adverse pressure gradient on the log law can be taken into account with the following *defect law*, as proposed by Perry and Schofield (1973):

$$\frac{U_e - U}{U_s} = f_2(\eta_2) = 1 - 0.4\eta_2^{1/2} - 0.6sin(\pi\eta_2/2)$$
(1.1)

where U_e is the edge velocity, $\eta_2 = y/\Delta$ and $\Delta = (\delta_1/C_s)(U_e/U_s)$. U_s is determined by fitting the data of eqn. 1.1 and $U_s/U_{MP} = 8(\Delta/L)^{1/2}$. L is the distance from the wall to the maximum in the shear stress profile; U_{MP}^2 is the maximum shear stress and C_s is a universal constant found empirically to be 0.35.

However, even if the above correlation applies to a wide range of adverse pressure gradients case, some restrictions $(-\rho \overline{u} v_M > 1.5 \text{ and low curvature surface})$ apply. Moreover, the log law and the defect law are valid upstream the ID point and approximatively valid upstream the ITD. A general equation valid for all the possible cases is today not available.

1.2.3 The SBLI code

The SBLI code is a compressible Navier-Stakes equations solver based on the finite difference methodology. It has been developed mainly by Dr. Yufeng Yao during his post doctoral position at Southampton University under the supervision of Prof. Neil D. Sandham. The code has been adopted by a large community of researchers spreads between Southampton University, Kingston University and Queen Mary University of London due to its large versatility and special features here briefly described (see Chapter 3 for a detailed presentation).

The spatial discretizion is based on a fourth order central scheme coupled with a conservative stable boundary scheme which satisfy the summation by part property (see Garritsen and Olsson, 1996, Olsson 1995a, Olsson 1995b and Carpenter *et al.*, 1999). An entropy splitting approach (see Sandham and Yee, 2000) has been implemented in order to improve the stability proprieties and non-reflecting boundary

conditions, based on the work of Poinsot and Lele (1992), are applied at inlet and outlet of the domain. The shock capturing is performed by a total variation diminishing (TVD) scheme based on the artificial compression method (ACM) described in Yee *et al.* (1999). The solver can also be used for large eddy simulation, with the following subgrid models: Smagorisky, dynamic Smagorinsky and Nagano models. The MPI/Fortran library has been used for the parallelization of the code.

Successful works have been carried out with the SBLI code in the past years, here is a brief resume. A numerical study of the Mach number effect on the compressible wall bounded turbulence was carried out by Li (PhD thesis, 2003). Here, the DNS of a turbulent channel at different grid resolutions has been carried out in order to identify a set of compressibility parameters able to describe the effect of the intrinsic compressibility on isothermal-wall channel flow. Moreover, in the same work an oblique shock-boundary layer interaction at free stream Mach number $M_{\infty} = 2$ was investigated. The capability of the code to deal with high Mach number supersonic flows has been successfully tested in the PhD work of Narasimhan (2005), where the investigation of the dynamics of turbulent spots in compressible flows up to Mach 6 were carried out. In the work of Sandham et al. (2003) large eddy simulations of transonic flow over a bump were carried out and a new synthetic approach to generate turbulent boundary layers was successfully implemented. DNS works on the boundary layer separation over an airfoil profile have been recently carried out by Jones at al. (2008) and Sandberg and Sandham (2008) with a similar version of the SBLI code, but adapted to airfoil profiles. Considering the increasing number of users and then the need for a standard reference and complete version, a recent work of re-engineering of the code was presented in Yao et al. (2009).

1.3 Objectives of the Present Study and Approach

The main goal of the present work is the extension of the SBLI code to fully 3D curvilinear capability in order to perform direct numerical simulations of turbulent flows over complex geometries. The extended version can be used to simulate flows around bluff bodies, like irregular shapes, hills, etc. and then to study complex phenomena like the three-dimensional boundary layer separation.

Two main simulations have been carried out: I) the DNS of a turbulent flow in a ridged channel; II) the DNS of a turbulent flow over an axisymmetric hill. In both cases the objectives are to validate the fully 3D curvilinear version of the code and, at the same time, to study the flow dynamics of the two systems.

For the ridged channel, same flow conditions of previously published works have been used and a comparison of the results is presented. The main interest is in the effect of the ridge on the turbulent flow at different Reynolds numbers. Simulations at $Re_{\tau} = 150$ and $Re_{\tau} = 360$ have been carried out. Due to the compressible form of the SBLI code, a Mach number $M_{cl} \approx 1.5$ has been used in order to increase the computational time step and, then, to reduce the computational cost. A sketch of the ridged channel is given in Figure 1.3.

For the hill case, due to the high Reynolds numbers of the experiments, no previous direct numerical simulations have been carried out. The goal is to investigate the three-dimensional boundary layer separation which occurs behind the hill and to provide DNS data to use as benchmark for RANS and LES turbulent models. However, some simplifications were necessary: firstly, the Reynolds number is only 5% of the experimental one ($Re_H \simeq 130000$). Secondly, as for the ridged channel case, we have used a Mach number at centerline $M_{\infty}=0.6$ (based on bulk density and wall viscosity) in order to reduce the computational cost. A sketch of the axisymmetric hill is given in Figure 1.4.

A multiblock version of the SBLI code has been also proposed. The objective is to
allow the simulation of turbulent flow over very complex geometries like jets in cross flow, cavity problems, etc. A two block test case has been performed as demonstration purposes. It consist in a subsonic jet in a turbulent cross flow at different Reynolds numbers and different jet to cross flow velocity ratios. A comparison with previous experimental and numerical works is presented.

The main steps of the present work are summarized in the following points:

- Review of the previous works on the fully 3D curvilinear form of the Navier-Stokes equations and comparison with previous quasi-3D version of the SBLI code.
- Rewriting of all the equations for compressible flow in the fully 3D curvilinear form and implementation into the code.
- Code validation tests, like laminar flow in a plane channel and free stream preservation in highly distorted meshes.
- DNS of a turbulent flow in a ridged channel and comparison with previous experimental and numerical works.
- DNS of a flow over an axisymmetric hill and investigation on the mechanism of the three-dimensional boundary layer separation.
- Extension to multiblock version and DNS of a jet in cross flow as demonstration test case.

1.4 Outline of Thesis

In this chapter we have given a brief introduction on the CFD, on the turbulence challenges and on the DNS technique. The main objectives of this work and a brief



Figure 1.3: Sketch of the ridged channel.



Figure 1.4: Sketch of the hill system.

literature review have also been presented. In the rest of the thesis, the chapters are organized as follows: Chapter 2 is dedicated to the governing equations implemented into the SBLI code and to the mathematical manipulations used to obtain the fully 3D curvilinear formulation. The numerical methods applied to those equations are given in Chapter 3, while Chapter 4 concerns itself with the code validation tests carried out for the fully 3D curvilinear version. In Chapter 5 the results obtained from the DNS of a turbulent flow in a ridged channel are presented, while Chapter 6 is dedicated to a turbulent flow over an axisymmetric hill. Chapter 7 concerns about the extension of the present code to a multiblock version. The DNS of a jet in a turbulent cross flow has been performed as demonstration test case. Finally, the main findings and results are summarized in Chapter 8. In Appendices A-D the details on the governing equations, non reflecting boundary conditions and inlet conditions for generating a turbulent boundary layer are given.

Chapter 2

Governing Equations

In this chapter we present the equations of the fluid dynamics and how these equations are rewritten in the fully 3D curvilinear form. Section 2.1 presents the dimensional form of the equations and the procedure to obtain the non-dimensional form. This operation gives two main advantages: a) to identify a set of dimensionless variables which describe the fluid flow physics; b) to obtain values around the unity. Section 2.2 gives the hyperbolic form of the fluid dynamics equations necessary to apply the bodyfitted coordinate method and, then, to simulate flows around complex geometries. The main mathematical manipulations are here given (see Appendixes B and C for more details) and a comparison with the previous quasi-3D version of the code is presented. A brief summary in Section 2.3 closes this chapter.

2.1 3D Navier-Stokes Equations in Cartesian Coordinates

The equations solved in the SBLI code are the 3D compressible Navier-Stokes equations of the fluid dynamics. The perfect gas law equation closes the number of equations and number of independent variables problem (six equations for six variables). Moreover, a Newtonian fluid is assumed and, then, the correlation between stress and velocity is well defined. Constant parameters are used for the thermal proprieties (like the thermal conductivity k, the heat capacity C_p , etc.), while the viscosity μ is assumed to be a function of the temperature (e.g. the power law $\mu^* = T^{*0.76}$ for the air).

2.1.1 Dimensional Equations

In the following, the asterisk superscript * indicates dimensional variables. Constant proprieties do not have the asterisk because their non-dimensional values is unitary. The 3D compressible fluid dynamics equations in Cartesian coordinates and in differential dimensional form are:

continuity

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial (\rho^* u_i^*)}{\partial x_i^*} = 0, \qquad (2.1)$$

momentum

$$\frac{\partial(\rho^* u_i^*)}{\partial t^*} + \frac{\partial(\rho^* u_i^* u_j^* + p^* \delta_{ij})}{\partial x_j^*} - \frac{\partial \tau_{ij}^*}{\partial x_j^*} = 0, \qquad (2.2)$$

energy

$$\frac{\partial E_{tot}^*}{\partial t^*} + \frac{\partial [(E_{tot}^* + p^*)u_i^*]}{\partial x_i^*} + \frac{\partial q_i^*}{\partial x_i^*} - \frac{\partial (\tau_{ij}^* u_j^*)}{\partial x_i^*} = 0, \qquad (2.3)$$

where, for i, j = 1, 2, 3, we obtain the spatial coordinates (x^*, y^*, z^*) , the velocity components (u^*, v^*, w^*) and the thermal flux terms (q_x^*, q_y^*, q_z^*) . The density ρ^* and the pressure p^* are linked by the equation of state for a perfect gas:

$$p^* = \rho^* R T^*, \tag{2.4}$$

where R is the universal gas constant. For a Newtonian fluid, the stress components

 τ_{ij}^* are linked to velocity field by:

$$\tau_{ij}^* = \mu^* \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} - \frac{2}{3} \frac{\partial u_k^*}{\partial x_k^*} \delta_{ij} \right), \qquad (2.5)$$

where μ^* is the viscosity, δ_{ij} is the Kronecker's index and k = 1, 2, 3. For the transport of thermal conductivity energy we assume valid the Fourier's law:

$$q_i^* = -k \frac{\partial T^*}{\partial x_i^*},\tag{2.6}$$

Finally, the total energy E_{tot}^* is linked to internal energy e^* and to kinematic energy by:

$$E_{tot}^* = \rho^* (e^* + \frac{1}{2}u_i^* u_i^*), \qquad (2.7)$$

Another useful equation is the definition of Mach number for a perfect gas:

$$M = U^* / \sqrt{\gamma R T^*}, \tag{2.8}$$

where γ is the specific heat ratio, or adiabatic index, and U^* is the norma of the vector velocity:

$$U^* = \|u_i^*\|,\tag{2.9}$$

Note: as the Mach number M is already a non-dimensional quantity like the specific heat ratio γ , the asterisk is not used for this variable.

2.1.2 Derivation of Dimensionless Equations

The non-dimensional form of the above equations helps to reduce the number of variables of a generic physical phenomenon. It is based on the *Buckingham* Π -theorem (see any good text book of fluid dynamics, like Denn, 1980) that states that the number n of physical variables can be reduced to k = n - m, where m is the number of independent fundamental physical quantities. In order to recognize the typical dimensionless groups, it is necessary to define some reference values. In our case, we define the following dimensionless quantity where the pedis "r" indicates the reference conditions:

$$u_{i} = \frac{u_{i}^{*}}{U_{r}^{*}}, \qquad \rho = \frac{\rho^{*}}{\rho_{r}^{*}}, \qquad p = \frac{p^{*}}{\rho_{r}^{*}U_{r}^{*2}},$$
$$T = \frac{T^{*}}{T_{r}^{*}}, \qquad \mu = \frac{\mu^{*}}{\mu_{r}^{*}}, \qquad e = \frac{e^{*}}{U_{r}^{*2}},$$
$$t = \frac{t^{*}}{\frac{\delta_{r}^{*}}{U_{r}^{*}}} = \frac{t^{*}U_{r}^{*}}{\delta_{r}^{*}}, \qquad x_{i} = \frac{x_{i}^{*}}{\delta_{r}^{*}}.$$
(2.10)

It is possible now to transform the eqns. (2.1)-(2.3) in the dimensionless form. After several manipulations (details are given in Appendix A), we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{(\partial \rho u_i)}{\partial x_i} = 0, \qquad (2.11)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j + p\delta_{ij})}{\partial x_j} - \frac{\partial\tau_{ij}}{\partial x_j} = 0, \qquad (2.12)$$

$$\frac{\partial E_{tot}}{\partial t} + \frac{\partial [(E_{tot} + p)u_i]}{\partial x_i} + \frac{\partial q_i}{\partial x_i} - \frac{\partial u_j \tau_{ij}}{\partial x_i} = 0, \qquad (2.13)$$

The dimensionless groups are included in the stress tensor term τ_{ij} and in the flux term q_i as following:

$$\tau_{ij} = \frac{\mu}{Re} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right), \qquad (2.14)$$

and

$$q_i = -\frac{1}{Re} \frac{1}{Pr} \frac{\mu}{M^2(\gamma - 1)} \frac{\partial T}{\partial x_i},$$
(2.15)

where the Reynolds (Re), Mach (M) and Prandtl (Pr) number are defined as:

$$Re = \frac{\rho_r^* U_r^* \delta_r^*}{\mu_r^*}, \qquad M = \frac{U_r^*}{\sqrt{\gamma R T_r^*}}, \qquad Pr = \frac{C_p \mu_r^*}{k}.$$
 (2.16)

The choice of the reference values depend on the flow system that we want to simulate. For example, in a channel case it is convenient to take as reference the values at wall, like the wall temperature T_w , in case of isothermal walls, and the *friction velocity* (u_τ^*) in case of turbulent flow:

$$u_{\tau}^{*} = \sqrt{\frac{1}{\rho_{b}^{*}}\bar{\tau}_{w}^{*}} = \sqrt{\frac{\mu_{w}^{*}}{\rho_{b}^{*}}} \left(\frac{\partial \bar{u}^{*}}{\partial y^{*}}\Big|_{w}\right), \qquad (2.17)$$

where the overbar indicates the Reynolds averaged value. Half height of the channel (h) is usually used as characteristic length and the bulk density (defined as $\rho_b^* = 1/(2h) \int_{-h}^{+h} \widetilde{\rho^*} dy^*$, where the tilde indicates the Favre averaged value) is taken as the reference density. For the isothermal walls the viscosity at wall is also constant and can be taken as reference (μ_w^*) .

In a boundary layer flow the reference quantities are usually the free stream values indicated with a " ∞ " pedis: u_{∞}^* , ρ_{∞}^* , μ_{∞}^* and T_{∞}^* . As reference length is normally

chosen the displacement boundary layer thickness at the inlet of the system (δ_{in}^*) .

2.2 3D Navier-Stokes Equations in Curvilinear Coordinates

The use of body-fitted curvilinear method needs of a hyperbolic form of the fluid dynamics equations (see Anderson, 1995). In particular, eqns. (2.1)-(2.3) can be rewritten as:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0, \qquad (2.18)$$

where:

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_{tot}, \end{pmatrix}, \qquad (2.19)$$

$$F = \begin{pmatrix} \rho u \\ \rho u u + p - \tau_{xx} \\ \rho u v - \tau_{xy} \\ \rho u w - \tau_{xz} \\ (E_{tot} + p)u + q_x - \tau_{xx}u - \tau_{xy}v - \tau_{xz}w \end{pmatrix},$$
(2.20)

$$G = \begin{pmatrix} \rho v \\ \rho v u - \tau_{yx} \\ \rho v v + p - \tau_{yy} \\ \rho v w - \tau_{yz} \\ (E_{tot} + p)v + q_y - \tau_{yx}u - \tau_{yy}v - \tau_{yz}w \end{pmatrix}, \qquad (2.21)$$

$$H = \begin{pmatrix} \rho w \\ \rho w u - \tau_{zx} \\ \rho w v - \tau_{zy} \\ \rho w w + p - \tau_{zz} \\ (E_{tot} + p)w + q_z - \tau_{zx}u - \tau_{zy}v - \tau_{zz}w \end{pmatrix}, \qquad (2.22)$$

The method consists in the transformation from a generic physical domain (x, y, z) to an orthogonal computational domain (ξ, η, ζ) where the points are equidistant and compact differential schemes can be easier applied. This transformation is expressed by the relationships between the physical and the computational coordinates:

$$\xi = \xi(x, y, z),$$

$$\eta = \eta(x, y, z),$$

$$\zeta = \zeta(x, y, z).$$

(2.23)

The eqn. (2.18) can be transformed in the computational domain with the following manipulation:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial F}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial G}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial G}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial G}{\partial \zeta} \frac{\partial \zeta}{\partial y} + \frac{\partial H}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial H}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial H}{\partial \zeta} \frac{\partial \zeta}{\partial z} = 0.$$
(2.24)

Then, it is necessary to find the relationships between the derivatives of the computational coordinates $(\partial \xi / \partial x, \partial \eta / \partial x, \partial \zeta / \partial x, etc.)$ and the derivatives of the physical coordinates $(\partial x / \partial \xi, \partial y / \partial \xi, \partial z / \partial \xi, etc.)$, called *direct metrics* and *inverse metrics*, respectively. The equations (2.23), or direct transformations, are written in differential form:

$$d\xi = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy + \frac{\partial \xi}{\partial z}dz,$$

$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial z}dz,$$

$$d\zeta = \frac{\partial \zeta}{\partial x}dx + \frac{\partial \zeta}{\partial y}dy + \frac{\partial \zeta}{\partial z}dz,$$

(2.25)

or in matrix form:

$$\begin{pmatrix} d\xi \\ d\eta \\ d\zeta \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} & \frac{\partial\xi}{\partial z} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} & \frac{\partial\eta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$
 (2.26)

Same manipulation can be done for the inverse metrics using the inverse transformations:

$$x = x(\xi, \eta, \zeta),$$

$$y = y(\xi, \eta, \zeta),$$

$$z = z(\xi, \eta, \zeta),$$

(2.27)

to have:

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta,$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta,$$

$$dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta,$$

(2.28)

or in matrix form:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \\ d\zeta \end{pmatrix}.$$
 (2.29)

By comparing the eqn. (2.26) with the eqn. (2.29), we have:

$$\begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} & \frac{\partial\xi}{\partial z} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} & \frac{\partial\eta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{pmatrix}^{-1}.$$
(2.30)

Then, we can write the relationship between the two metrics using the inverse formula

of a matrix. The matrix on the right hand side of eqn. (2.30) is called Jacobian matrix and its determinant is indicated with J:

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix}.$$
 (2.31)

Finally, the relationship between the two metrics are:

$$\frac{\partial\xi}{\partial x} = \frac{1}{J} \left(\frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \eta} \right), \quad \frac{\partial\xi}{\partial y} = -\frac{1}{J} \left(\frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial \eta} \right), \quad \frac{\partial\xi}{\partial z} = \frac{1}{J} \left(\frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial \eta} \right), \\ \frac{\partial\eta}{\partial x} = -\frac{1}{J} \left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \xi} \right), \quad \frac{\partial\eta}{\partial y} = \frac{1}{J} \left(\frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \zeta} - \frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial \xi} \right), \quad \frac{\partial\eta}{\partial z} = -\frac{1}{J} \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial \xi} \right), \\ \frac{\partial\zeta}{\partial x} = \frac{1}{J} \left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \xi} \right), \quad \frac{\partial\zeta}{\partial y} = -\frac{1}{J} \left(\frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \xi} \right), \quad \frac{\partial\zeta}{\partial z} = \frac{1}{J} \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \right).$$

$$(2.32)$$

After several mathematical	manipulations	(details	are given	in A	Appendix	В),	it	is
possible to rewrite the eqn.	(2.24) in terms	of the co	omputatio	nal d	lomain:			

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial \xi} + \frac{\partial G_1}{\partial \eta} + \frac{\partial H_1}{\partial \zeta} = 0, \qquad (2.33)$$

where:

$$U_1 = JU,$$

Chapter 2. Governing Equations

$$F_{1} = \left(JF\frac{\partial\xi}{\partial x} + JG\frac{\partial\xi}{\partial y} + JH\frac{\partial\xi}{\partial z}\right),$$

$$G_{1} = \left(JF\frac{\partial\eta}{\partial x} + JG\frac{\partial\eta}{\partial y} + JH\frac{\partial\eta}{\partial z}\right),$$

$$H_{1} = \left(JF\frac{\partial\zeta}{\partial x} + JG\frac{\partial\zeta}{\partial y} + JH\frac{\partial\zeta}{\partial z}\right).$$

The eqn. (2.33) is the equation solved in the SBLI code in the computational domain (ξ, η, ζ) . In the previous version of the SBLI code (see Sandham *et al.*, 2003), only geometries with a linear stretching in the *z*-coordinate were possible to solve. This is equivalent to pose equal zero the following values of the Jacobian matrix:

$$\frac{\partial x}{\partial \zeta} = \frac{\partial y}{\partial \zeta} = \frac{\partial z}{\partial \xi} = \frac{\partial z}{\partial \eta} = 0.$$

This leads to a simplification of the metrics relationships as:

$$\frac{\partial \xi}{\partial x} = \frac{1}{J} \frac{\partial y}{\partial \eta}, \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{J} \frac{\partial x}{\partial \eta},$$
$$\frac{\partial \eta}{\partial x} = -\frac{1}{J} \frac{\partial y}{\partial \xi}, \qquad \frac{\partial \eta}{\partial y} = -\frac{1}{J} \frac{\partial x}{\partial \xi},$$

(2.34)

and

$$\frac{\partial \zeta}{\partial z} \neq 0$$

A complete review of all the subroutines where the metric quantities are involved has been necessary. In the Appendix C are presented the mathematical manipulations and the new nomenclature used in the equations for the fully-3D curvilinear version of the code.

2.3 Summary

The dimensional and non-dimensional equations solved by the SBLI code are presented. Details about the reference values used in the channel and boundary layer flow are given and a presentation of the body-fitted coordinate method, accomplished with details of equations derivation, has been presented. All the equations are now ready to be discretized with the numerical method presented in the next chapter.

Chapter 3

Numerical Methods

In this chapter we describe the numerical methods used to solve the fluid dynamics equations presented in Chapter 2. Some descriptions of the numerical features of the SBLI code, like the TVD scheme and LES models (see Narasimhan 2005 and Sandham *et al.* 2003), are here omitted because, even if updated to the fully 3D curvilinear status, they have not been fully tested.

Section 3.1 describes the spatial discretization operator with an introduction on the stable treatment of the boundary conditions. Then, it follows the description of the time marching scheme (Section 3.2) and the relationship of stability between the time step and the CFL number. The entropy splitting method is introduced in Section 3.3 with a brief explanation of the energy estimation concept. The treatment of the boundary conditions by the characteristic form of the N-S equations is presented in Section 3.4. The integral condition is also defined. Section 3.5 concludes this chapter, where the filtering schemes used in presence of non smooth meshes is described.

3.1 Spatial Discretization

A fourth-order central scheme has been used for the spatial approximation of the derivatives terms. The first and second derivative operators for the internal and periodic boundary points, are:

$$f'_{m} = \frac{-f_{m+2} + 8f_{m+1} - 8f_{m-1} + f_{m-2}}{12\Delta h},$$
(3.1)

$$f_m'' = \frac{-f_{m+2} + 16f_{m+1} - 30f_m + 16f_{m-1} - f_{m-2}}{12\Delta h^2},$$
(3.2)

where Δh is the grid spacing.

In the presence of non periodic conditions (like a no slip wall boundary), the scheme use a one-side formula which is consistent with the central scheme. Moreover, the scheme needs to be conservative, in order to respect the fluid dynamics principles, and give a stable solution. A scheme proposed by Carpenter *et al.* (1999) satisfies these requirements and a first-derivative operator for the combined interior and boundary scheme is given by:

$$D\vec{u} = \frac{1}{\Delta h} P^{-1} Q\vec{u}, \qquad (3.3)$$

where $\vec{u} = [u_0, ..., u_N]^T$. The values of the matrix P and Q for a fourth order scheme are:





where a and b are coefficients with following value:

$$a = \frac{-(2117\sqrt{295369} - 1166427)}{25488},$$

$$b = \frac{66195\sqrt{53}\sqrt{5573} - 35909375)}{101952},$$
(3.4)
(3.5)

and the dots indicate continuation of previous entries along the matrix diagonal. The values of the matrix first derivative operator D is:

$$D = \frac{1}{\Delta h} \begin{bmatrix} -1.833 & 3.0 & -1.5 & 0.3334 \\ -0.3763 & -0.3225 & 0.7194 & 0.0394 & -0.0658 & 0.0057 \\ 0.1134 & -0.7913 & 0.1972 & 0.5214 & -0.0367 & -0.0041 \\ 0.0093 & -0.1219 & -0.7278 & 0.0451 & -0.6521 & -0.0820 \\ & & -0.0833 & 0.6667 & 0 & -0.6667 & 0.0833 \end{bmatrix}.$$

and the second derivative operator D'':

3.2 Time Marching Scheme

A third-order compact storage Runge-Kutta scheme is implemented in the SBLI code in order to integrate the fluid dynamics equations in the time (see Wray, 1986 and Spalart, 1991). The method require two storage locations U_A and U_B , updated as following:

I) Step one

$$t = t_n,$$

$$U_B = U_A,$$

$$U_A^1 = U_A + a_{11} \Delta t f(U_A, t_n),$$

$$U_B^1 = U_B + a_{21} \Delta t f(U_A, t_n).$$

II) Step two:

$$t^{1} = t_{n} + a_{11}\Delta t,$$

$$U_{A}^{2} = U_{B}^{1} + a_{12}\Delta t f(U_{A}^{1}, t_{n}^{1}),$$

$$U_{B}^{2} = U_{B}^{1} + a_{22}\Delta t f(U_{A}^{1}, t_{n}^{1}).$$

III) Step three:

$$t^{2} = t_{n} + (a_{12} + a_{21})\Delta t,$$

$$U_{A}^{3} = U_{B}^{2} + a_{13}\Delta t f(U_{A}^{2}, t_{n}^{2}),$$

$$U_{B}^{3} = U_{B}^{2} + a_{23}\Delta t f(U_{A}^{2}, t_{n}^{2}).$$

IV) Step four:

$$t_{n+1} = t_n + \Delta t,$$
$$U_A = U_B^3,$$
$$U_B = U_B^3.$$

where a_{1j} and a_{2j} are constants for the three sub time step (j = 1, 2, 3):

$$a_{1j} = (2/3, 5/12, 3/5), \qquad a_{2j} = (1/4, 3/20, 3/5).$$
 (3.6)

In general, the stability limit imposes the following relationship between the time step and the CFL number:

$$\Delta t = \frac{CFL}{|b|_{max}},\tag{3.7}$$

where the maximum CFL value is fixed by the time-advance method, which for the RK3 is $\sqrt{3}$. The value of b for a plane channel is:

$$b = \frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + \frac{|w|}{\Delta z} + f_1 \sqrt{\frac{\gamma p}{\rho}} + 4f_2 \left(\frac{\gamma}{RePr}\right), \qquad (3.8)$$

where

$$f_1 = \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}},$$
(3.9)

$$f_2 = \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} + \frac{2}{\Delta x \Delta y} + \frac{2}{\Delta x \Delta z} + \frac{2}{\Delta y \Delta z}.$$
(3.10)

For flows considered incompressible (M < 0.3) the time step become very small and an expensive computational time is normally required. This is a typical inconvenient of solvers for compressible fluids used to simulate flows at very low Mach number.

For the boundary layer flow the following formula is adopted:

$$\Delta t = \frac{CFL}{\Delta c + \Delta \mu},\tag{3.11}$$

where

$$\Delta c = \pi \frac{\sqrt{T}}{M} \left(\frac{1}{\Delta x} + \frac{1}{\Delta y} + \frac{1}{\Delta z} \right), \qquad (3.12)$$
$$\Delta \mu = \frac{\pi^2 \mu}{(\gamma - 1)M^2 RePr\rho} \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right].$$

This formula is obtained by assuming periodic conditions in all the direction and solving a Fourier transformed convection-diffusion equation model.

3.3 Entropy Splitting

Turbulent flow simulation requires long integration time which normally can indefinitely increase the numerical error causing the instability of the solution. As presented in Yee *et al.* (1999), the application of the energy estimation method to the Euler equations (see the work of Harten, 1983, Olsson, 1995 part I and part II and Gerritsen and Olsson, 1996) can be applied to the Navier-Stokes equations in order to stabilize the solution of long time integration problems. The basic idea consists of a transformation of the equations in a symmetric form by an adapted variable w and then splitting, by a splitting parameter β , in a conservative portion and in another non-conservative portion, both in symmetric form. Then, the summation by parts propriety can be applied to each portion in order to estimate an upper bound to the energy growth. This will guarantee the stability of the algorithm.

The procedure can be explained considering the 1D-convective equation:

$$u_t + f_x = 0, (3.13)$$

where the variable of derivation has been indicated with the pedis. Applying the chain rule with the above mentioned variable w we obtain:

$$u_t + f_x = u_w w_t + f_w w_x = 0, (3.14)$$

where the Jacobian of f_w is symmetric, u_w is symmetric and positive defined and u(w) and f(w) are homogeneous functions of w, i.e.:

$$u(\theta w) = \theta^{\beta} u(w), \qquad (3.15)$$

$$f(\theta w) = \theta^{\beta} f(w), \qquad \theta, \beta \in \Re$$
(3.16)

Moreover, they satisfy the Euler's differential equation:

$$u_w w = \beta u, \tag{3.17}$$

$$f_w w = \beta f. \tag{3.18}$$

We apply now the canonical splitting to the transformed derivative flux vector $f_w w_x$:

$$(1+\beta)u_t = \beta f_x + \beta f_w w_x, \qquad (3.19)$$

$$f_w w_x = \frac{\beta}{1+\beta} f_x + \frac{1}{\beta+1} f_w w_x = 0, \qquad (3.20)$$

and, applying the scalar product between w and each term of the previous equation, we have:

$$(1+\beta)(w, u_t) = \beta(w, f_x) + \beta(w, f_w w_x).$$
(3.21)

Using the integration by parts theorem:

$$(f_w w, w_x) = w^T f_w w|_a^b - ((f_w w)_x, w).$$
(3.22)

The above equation can be rewritten using the propriety defined by eqns. (3.15-3.18):

$$\beta(w, f_x) = (w, (\beta f)_x) = (w, (f_w w)_x) = ((f_w w)_x, w),$$
(3.23)

and, if we substitute in eqn. (3.13), we have:

$$(1+\beta)u_t = ((f_w w)_x, w) + w^T f_w w|_a^b - ((f_w w)_x, w) = w^T f_w w|_a^b.$$
(3.24)

We can demonstrate that this last equation is equal to an estimate of the energy:

$$\frac{d}{dt}(w, u_w w) = (w_t, u_w) + (w, (u_w w)_t) = (u_t, w) + \beta(w, u_t) = (1 + \beta)(w, u_t) = -w^T f_w w|_a^b.$$
(3.25)

We conclude that the split form give an estimate of the energy growth bounded between the boundary condition in a and b.

Applied to the Navier-Stokes equations, the entropy splitting form can be written as:

$$U_t + \frac{\beta}{\beta+1}(F_x + G_y + H_z) + \frac{1}{\beta+1}(F_W W_x + G_W W_y + H_W W_z) = \frac{1}{Re}(F_x^v + G_y^v + H_z^v),$$
(3.26)

with $\beta \neq -1$ and

$$W = [w_1, w_2, w_3, w_4, w_5] = p^{\dagger} / p[E_{tot} + (\alpha - 1)/(\gamma - 1)p, -\rho u, -\rho v, -\rho w, \rho]^T.$$
(3.27)

The upper triangular part of the symmetric matrices F_W , G_W and H_W are given by

$$F_{W} = \frac{1}{p^{\dagger}} \begin{bmatrix} c_{1}\rho u & c_{1}\rho u^{2} - p & c_{1}\rho uv & c_{1}\rho uw & u[c_{1}E_{tot} + (c_{2} - 1)p] \\ u(c_{1}\rho u^{2} - 3p) & v(c_{1}\rho u^{2} - p) & w(c_{1}\rho u^{2} - p) & u^{2}[c_{3} - \frac{p}{\rho}(E_{tot} + p)] \\ u(c_{1}\rho v^{2} - p) & c_{1}\rho uvw & uv[c_{1}E_{tot} + (c_{2} - 2)p] \\ u(c_{1}\rho w^{2} - p) & uw[c_{1}E_{tot} + (c_{2} - 2)p] \\ uc_{4} \end{bmatrix}$$

$$(3.28)$$

$$G_{W} = \frac{1}{p^{\dagger}} \begin{bmatrix} c_{1}\rho v & c_{1}\rho uv & c_{1}\rho v^{2} - p & c_{1}\rho vw & v[c_{1}E_{tot} + (c_{2} - 1)p] \\ v(c_{1}\rho u^{2} - p) & u(c_{1}\rho v^{2} - p) & c_{1}\rho uvw & uv[c_{1}E_{tot} + (c_{2} - 2)p] \\ v(c_{1}\rho v^{2} - 3p) & w(c_{1}\rho v^{2} - p) & v^{2}[c_{3} - \frac{p}{\rho}(E_{tot} + p)] \\ v(c_{1}\rho w^{2} - p) & vw[c_{1}E_{tot} + (c_{2} - 2)p] \\ vc_{4} \end{bmatrix}$$

$$(3.29)$$

,

|,

$$H_{W} = \frac{1}{p^{\dagger}} \begin{vmatrix} c_{1}\rho w & c_{1}\rho uw & c_{1}\rho vw & c_{1}\rho w^{2} - p & w[c_{1}E_{tot} + (c_{2} - 1)p] \\ w(c_{1}\rho u^{2} - p) & c_{1}\rho uvw & u(c_{1}\rho w^{2} - p) & uw[c_{1}E_{tot} + (c_{2} - 2)p] \\ w(c_{1}\rho v^{2} - p) & v(c_{1}\rho w^{2} - p) & vw[c_{1}E_{tot} + (c_{2} - 2)p] \\ w(c_{1}\rho w^{2} - 3p) & w^{2}[c_{3} - \frac{p}{\rho}(E_{tot} + p)] \\ uc_{4} \\ (3.30) \end{vmatrix}$$

where

$$p^{\dagger} = -(p\rho^{-}\gamma)^{\frac{1}{\beta(1-\gamma)}},\tag{3.31}$$

$$c_1 = \frac{1 - \beta(1 - \gamma)}{\beta(1 - \gamma) - \gamma},\tag{3.32}$$

$$c_2 = \frac{1}{\beta(1-\gamma) - \gamma},\tag{3.33}$$

$$c_3 = c_1 E_{tot} + (c_2 - 2)p, (3.34)$$

$$c_4 = \frac{c_1 E_{tot}^2}{\rho} + p[2(c_2 - 1)\frac{E_{tot}}{\rho} - q^2] + \frac{p^2}{\rho}[c_2(1 + \beta) - 2], \qquad (3.35)$$

$$q^2 = u^2 + v^2 + w^2. aga{3.36}$$

$$F^{v}, G^{v}$$
 and H^{v} are the viscous and heat conduction terms, with

$$F^{v} = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + \frac{\mu}{(\gamma-1)PrM^{2}} \frac{\partial T}{\partial x} \end{bmatrix} = \begin{bmatrix} \mu \frac{\partial \rho}{\partial x} \\ \mu \frac{\partial u}{\partial x} \\ \mu \frac{\partial u}{\partial x} \\ \mu \frac{\partial w}{\partial x} \\ \mu \frac{\partial w}{\partial x} \\ \mu \frac{\partial A}{\partial x} \end{bmatrix} + \begin{bmatrix} -\mu \frac{\partial \rho}{\partial x} \\ \mu (\frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial u_{l}}{\partial x_{l}}) \\ \mu \frac{\partial u}{\partial y} \\ \mu \frac{\partial u}{\partial z} \\ B_{1} \end{bmatrix}, \quad (3.37)$$

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where $A = \gamma/\Pr(E_{tot}/\rho - \frac{1}{2}u_iu_i) + \frac{1}{2}u_iu_i$, $B_1 = \mu u(\partial u/\partial x - \frac{2}{3}\partial u_l/\partial x_l) + \mu v \partial u \partial y + \mu w \partial u \partial z$, $B_2 = \mu u(\partial v/\partial x) + \mu v(\partial v/\partial y - \frac{2}{3}\partial u_l/\partial x_l) + \mu w \partial v \partial z$, $B_3 = \mu u(\partial w/\partial x) + \mu v(\partial w/\partial y) + \mu w(\partial w/\partial z - \frac{2}{3}\partial u_l/\partial x_l)$.

3.4 Characteristic Non-Reflecting Boundary Conditions

To properly simulate the incoming and outgoing fluid flow at the inlet/outlet of the domain, the characteristic non-reflecting boundary conditions method, based on the work of Poinsot and Lele (1992) and Strikwerda (1977), has been used. The idea is based on the characteristic form of the Euler equations, extended to the Navier-Stokes equations, where the incoming waves external to the computational domain are set equal zero. Considering the eqn. (2.18) in a compact form:

$$U_t + F_x + G_y + H_z = r.h.s. (3.40)$$

and introducing the non-conservative variables $\hat{U} = [\rho, u, v, w, p\rho^{\gamma}]^T$ and $V = [\rho, u, v, w, p]^T$, we obtain, using the chain rule:

$$U_t + RT^{-1}LTSV_x + R\hat{T}^{-1}M\hat{T}SV_y + R\tilde{T}^{-1}N\tilde{T}SV_z = r.h.s.,$$
(3.41)

where

$$R = \frac{\partial U}{\partial \hat{U}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ u & \rho & 0 & 0 & 0 \\ v & 0 & \rho & 0 & 0 \\ w & 0 & 0 & \rho & 0 \\ a & \rho u & \rho v & \rho w & \frac{\rho^{\gamma}}{\gamma - 1} \end{bmatrix},$$
 (3.42)

with $a = c^2/(\gamma - 1) + (u^2 + v^2 + w^2)/2$ and c the speed of sound. Moreover:

$$T^{-1} = \begin{bmatrix} \frac{1}{2c^2} & \frac{-1}{c^2} & 0 & \frac{-1}{c^2} & 0\\ \frac{-1}{2\rho c} & 0 & 0 & 0\frac{1}{2\rho c} \\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & \frac{1}{\rho^{\gamma}} & 0 & 0 & 0 \end{bmatrix}, \quad LTSV_x = \begin{bmatrix} (u-c) \left[\frac{dp}{dx} - \rho c\frac{du}{dx} \right] \\ u \left[\frac{dp}{dx} - c^2 \frac{d\rho}{dx} \right] \\ u \frac{dw}{dx} \\ u \frac{dw}{dx} \\ (u+c) \left[\frac{dp}{dx} + \rho c\frac{du}{dx} \right] \end{bmatrix}, \quad (3.43)$$

$$\hat{T}^{-1} = \begin{bmatrix} \frac{1}{2c^2} & 0 & \frac{-1}{c^2} & 0 & \frac{-1}{c^2} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{-1}{2\rho c} & 0 & 0 & 0\frac{1}{2\rho c} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\rho^{\gamma}} & 0 & 0 \end{bmatrix}, \quad M\hat{T}SV_y = \begin{bmatrix} (v-c) \left[\frac{dp}{dy} - \rho c\frac{dv}{dy}\right] \\ v \left[\frac{dp}{dy} - c^2\frac{d\rho}{dy}\right] \\ v \left[\frac{dp}{dy} - c^2\frac{d\rho}{dy}\right] \\ (v+c) \left[\frac{dp}{dy} + \rho c\frac{dv}{dy}\right] \end{bmatrix}, \quad (3.44)$$

$$\tilde{T}^{-1} = \begin{bmatrix} \frac{1}{2c^2} & 0 & 0 & \frac{-1}{c^2} & \frac{-1}{c^2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{-1}{2\rho c} & 0 & 0 & 0\frac{1}{2\rho c} \\ 0 & 0 & 0 & \frac{1}{\rho^{\gamma}} & 0 \end{bmatrix}, \quad N\tilde{T}SV_z = \begin{bmatrix} (w-c) \left[\frac{dp}{dz} - \rho c\frac{dw}{dz}\right] \\ w\frac{du}{dz} \\ w\frac{dv}{dz} \\ w\left[\frac{dp}{dz} - c^2\frac{d\rho}{dz}\right] \\ (w+c) \left[\frac{dp}{dz} + \rho c\frac{dw}{dz}\right] \end{bmatrix}, \quad (3.45)$$

with $S = \partial \hat{U} / \partial V$, $T = (\partial F / \partial U)^{-1}$, $\hat{T} = (\partial G / \partial U)^{-1}$ and $\hat{T} = (\partial H / \partial U)^{-1}$.

The non-reflective nature of the boundary condition is based on the sign of the eigenvalue of the diagonal matrix L, M and N presented in eqns. (3.43)-(3.45) which represent the characteristic ingoing and outgoing into and from the domain. At the inlet, if the eigenvalue is positive (incoming wave) then the characteristic is set to zero, while if negative is left as is. Similar rule applies for the outlet boundary.

A good way to maintain the value of the initial flow at the inlet of the domain (usually located at x = 0) is to filter the outgoing waves by an integral method, that is: $U^{n+1} = U^n + \int_{t_n}^{t_{n+1}} (\partial F/\partial x) dt$, where F is the flux vector normal to the inlet. This condition is particularly efficient in applications where the inlet is quite far away from the obstacle, which can form ingoing waves (like a hill or a bluff body). If the distance is not large enough, the inlet condition can be drastically changed and an extrapolation of the pressure field can be a preferred solution. In this case, the value of the pressure at the inlet is found with the following one side formula:

$$p_1 = 2p_2 - p_3, \tag{3.46}$$

where p_i , for i = 1 - 3, is pressure in the first, second and third point, respectively.

3.5 Filtering Scheme

In applications where non-smooth meshes are involved, like in presence of an abrupt change of slope or a sudden mesh coarsening, spurious oscillations can be generated

filter order	a_0	a_1	a_2	a_3	
4th	$\frac{5}{8} + \frac{3\alpha_f}{4}$	$\frac{1}{2} + \alpha_f$	$\frac{-1}{8} + \frac{\alpha_f}{4}$	0	
6th	$\frac{11}{16} + \frac{5\alpha_f}{8}$	$\frac{15}{32} + \frac{17\alpha_f}{16}$	$\frac{-3}{16} + \frac{3\alpha_f}{8}$	$\frac{1}{32} - \frac{\alpha_f}{16}$	

Table 3.1: Coefficients for 4th- and 6th-order filtering scheme

and the correctness and stability of the whole simulation be inhibit. In these cases a filtering operation can be an easy solution of the problem without to drastically change the flow field of the unfiltered solution. The filter operator is the same defined in Visbal and Gaitonde (2001) and (2002):

$$\alpha_f \hat{\phi}_{i-1} + \hat{\phi}_i + \alpha_f \hat{\phi}_{i+1} = \sum_{n=0}^N \frac{a_n}{2} (\phi_{i+n} + \phi_{i-n}), \qquad (3.47)$$

where $\hat{\phi}$ is the filtered quantities and ϕ the unfiltered one. The coefficients for fourth and sixth-order schemes are presented in table 3.1. The value of the parameter $-0.5 \leq \alpha_f \leq 0.5$ controls the dissipation of the filter: at higher values correspond less dissipation and viceversa. For $\alpha_f = 0$ the scheme became explicit and some advantages are obtained: a) the matrix of the unknown filtered quantities is not tridiagonal and then a faster solution can be applied; b) if the computational domain is divided between more processors, an implicit scheme require a parallel tridiagonal solver which implies complicate communications between the processors. In the current version of the code the extreme terms of the tridiagonal matrix ($\hat{\phi}_{0-1}$ and $\hat{\phi}_{N+1}$) are considered known and equal to the unfiltered values ($\hat{\phi}_{0-1} = \phi_{0-1}$ and $\hat{\phi}_{N+1} = \phi_{N+1}$).

3.6 Summary

A description of the numerical methods implemented in the SBLI code has been given. The spatial derivative and temporal integration schemes have been described and a short introduction on the entropy splitting concept is also presented with an example on the 1D convective equation. The use of the entropy splitting associated to the special treatment of the boundary conditions gives a stable and conservative algorithm. Special attention has been also given to the treatment of the inflow/outflow conditions by use of characteristic non-reflecting boundary conditions. Finally, a filter operator used for non smooth meshes has been described and implemented into the code which is now ready to be tested as shown in the next chapter.

Chapter 4

Code Validation

Some numerical experiments have been carried out in order to validate the SBLI code with fully 3D curvilinear capability. These tests may not have a practical application, due to their simplified flow conditions, but they are often used as benchmark cases. Section 4.1 presents results on a free stream flow preservation on a wavy mesh. A different formulation of the inviscid terms of eqn. (2.33) is here proposed in order to avoid numerical errors commonly encountered in the fully 3D curvilinear formulation. A laminar flow in a plane channel with large grid distortions is presented in Section 4.2. The purpose is to verify the symmetrical propriety of the numerical errors obtained in the velocity components while the main flow direction and position of the walls are changed rotationally around the three physical coordinates. Finally, Section 4.4 presents the pre-compiler program used to simplify the r.h.s subroutine when flows in non curvilinear geometries are investigated.

4.1 Free Stream Preservation on a Wavy Grid

This test was proposed by Visbal and Gaitonde (2002) and it consists in the preservation of a uniform free stream flow on a wavy grid. Because no external forces are

applied on the fluid, the initial flow field should be preserved in the time and in the space, whatever is the mesh used to discretize the domain. An eventual departure from their initial value of the velocity components indicate an error in the implementation of the fully 3D curvilinear form. The mesh, presented in Figure 4.1, is described by the following formula as:

$$x = \overline{x} - \frac{L_x}{2} + A_x \overline{\Delta x} \sin\left(\frac{2\pi w_x}{L_x}\overline{x}\right) \sin\left(\frac{2\pi w_y}{L_y}\overline{y}\right) \sin\left(\frac{2\pi w_z}{L_z}\overline{z}\right),$$
(4.1)

$$y = \overline{y} - \frac{L_y}{2} + A_y \overline{\Delta y} \sin\left(\frac{2\pi w_x}{L_x}\overline{x}\right) \sin\left(\frac{2\pi w_y}{L_y}\overline{y}\right) \sin\left(\frac{2\pi w_z}{L_z}\overline{z}\right),$$
(4.2)

$$z = \overline{z} - \frac{L_z}{2} + A_z \overline{\Delta z} \sin\left(\frac{2\pi w_x}{L_x}\overline{x}\right) \sin\left(\frac{2\pi w_y}{L_y}\overline{y}\right) \sin\left(\frac{2\pi w_z}{L_z}\overline{z}\right),$$
(4.3)

where $\overline{x}, \overline{y}, \overline{z}$ are the x, y, z-coordinates of a uniform mesh, respectively (e.g. $\overline{x} = (i-1)\overline{\Delta x}$, etc., where *i* is the grid index and $\overline{\Delta x}$ the grid spacing in the *x*-direction) and A_i and w_i are, respectively, the amplitude and the frequency of the sinusoidal wave function, respectively.

As pointed out by Pulliam and Steger (1978), when a derivative operator of central scheme is applied with a 3D body-fitted curvilinear coordinate system, an error can occur due to the non cancelation of the following terms:

$$I_1 = \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) = (\hat{\xi}_x)_{\xi} + (\hat{\eta}_x)_{\eta} + (\hat{\zeta}_x)_{\zeta}, \tag{4.4}$$

$$I_2 = \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) = (\hat{\xi}_y)_{\xi} + (\hat{\eta}_y)_{\eta} + (\hat{\zeta}_y)_{\zeta}, \tag{4.5}$$



Figure 4.1: Wavy mesh used for the free stream preservation test.

$$I_3 = \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = (\hat{\xi}_z)_{\xi} + (\hat{\eta}_z)_{\eta} + (\hat{\zeta}_z)_{\zeta}.$$
(4.6)

where

$$\hat{\xi}_x = J \frac{\partial \xi}{\partial x}, \quad \hat{\eta}_x = J \frac{\partial \eta}{\partial x}, \quad \hat{\zeta}_x = J \frac{\partial \zeta}{\partial x}, \quad (4.7)$$

$$\hat{\xi}_y = J \frac{\partial \xi}{\partial y}, \quad \hat{\eta}_y = J \frac{\partial \eta}{\partial y}, \quad \hat{\zeta}_y = J \frac{\partial \zeta}{\partial y}, \quad (4.8)$$

$$\hat{\xi}_z = J \frac{\partial \xi}{\partial z}, \quad \hat{\eta}_z = J \frac{\partial \eta}{\partial z}, \quad \hat{\zeta}_z = J \frac{\partial \zeta}{\partial z}.$$
 (4.9)

Pulliam and Steger proposed an averaging procedure in order to guarantee the free stream preservation, but that approach is difficult to extend to high-order formulation. Thomas and Lombard (1978) suggested the following conservative form for the metric terms:

$$\begin{aligned} \xi_x &= y_\eta z_\zeta - y_\zeta z_\eta = (y_\eta z)_\zeta - (y_\zeta z)_\eta \,, \\ \xi_y &= z_\eta x_\zeta - z_\zeta x_\eta = (z_\eta x)_\zeta - (z_\zeta x)_\eta \,, \end{aligned}$$

$$\xi_z = x_\eta y_\zeta - x_\zeta y_\eta = (x_\eta y)_\zeta - (x_\zeta y)_\eta,$$

etc, ... (4.10)

It easy to verify that the above formulation satisfy the metric cancelation when a central scheme is applied. The method is easy to apply to high-order scheme and good results are obtained as shown in Visbal and Gaitonde (2002). However, a different but much simpler reformulation has been here introduced. The idea is to solve, for the inviscid terms, the eqn. B.1 without the changing form described in Appendix B. The same equation can be obtained expanding the derivative operator applied to the inviscid terms presented in Appendix C, if the I_1 , I_2 and I_3 terms are zero. For example, the ξ -derivatives of the x-momentum equation will be computed as:

$$\frac{\partial}{\partial\xi} \left[(\rho u u + p) J \frac{\partial\xi}{\partial x} \right] + \frac{\partial}{\partial\xi} \left[(\rho u v) J \frac{\partial\xi}{\partial y} \right] + \frac{\partial}{\partial\xi} \left[(\rho u w) J \frac{\partial\xi}{\partial z} \right] = \frac{\partial}{\partial\xi} \left[(\rho u u + p) \right] J \frac{\partial\xi}{\partial x} + \frac{\partial}{\partial\xi} \left[(\rho u v) \right] J \frac{\partial\xi}{\partial y} + \frac{\partial}{\partial\xi} \left[(\rho u w) \right] J \frac{\partial\xi}{\partial z}.$$
(4.11)

Following this approach we force the cancelation of I_1 - I_3 terms and the free stream preservation can be guaranteed even without the continuity and second derivability conditions that are necessary for the mathematical manipulations presented in Appendix B (these conditions are necessary to apply the Schwartz's theorem on the cross derivatives which simplifies the equations reported in Appendix B).

The free stream preservation test has been carried out with the following computational values: $L_x = L_y = L_z = 4$, $A_x = A_y = A_z = 1$ and $w_x = w_y = w_z = 1/4$ for a mesh of $N_x = N_y = N_z = 21$ points; Re = 500, M = 1, u = 1, v = w = 0 as flow conditions. After 100 iteration at a computational time step of 10^{-4} the departure from the initial value has been found around 10^{-16} .

4.2 Laminar Flow in a Plane Channel with large grid distortions

In the previous free stream preservation test all the metric terms were different from zero. However, due to the uniform value of the velocity field applied, many of those terms were multiplied by the null value of the velocity derivatives. A laminar flow in a plane channel with large grid distortions contains, at least, non-zero velocity derivatives in the wall normal direction. The mesh is stretched with a tangent hyperbolic function in the direction of the walls and with a cosinusoidal function, similar to the free stream case, in the other directions (see Figure 4.2).



Figure 4.2: Plane channel with large grid distortions.

The analytical solution is the well known parabolic Poiseuille profile, but here same difference from the theoretical values are quite evident due to the high distortions and coarse mesh resolution. However, these numerical errors are the key of our test: on a coarse and distorted mesh the numerical derivatives can be quite different from the correct analytical value and a relatively large error is introduced into the simulation. Due to the laminar nature of the flows (if the error does not grow up to have an unstable condition), by alternating the directions of the flow and the position of the walls, same results should be obtained. Because of the three-dimensional nature of the code, three pair of numerical errors are expected for a total of six tests. Table 4.1 gives the test conditions and maximum error in each direction. The simulations are carried out at Re = 20 and M = 0.1 with a fix time step of dt = 0.01 and for 100 iterations. The dimensions of the plane channel are $L_x = L_y = L_z = 2$, while 17 points in each direction have been used. Periodicity is applied except on the wall side and the flow is driven by a constant pressure gradient of $\partial p/\partial x = -1$. The entropy splitting was switched off using an entropy factor $\beta = 1 \times 10^{10}$ to avoid loss of mass. In each

Test	wall in x	wall in y	wall in z	$-\frac{\partial p}{\partial x}$	$-\frac{\partial p}{\partial y}$	$-\frac{\partial p}{\partial z}$	$ \rho u _{max}$	$ ho v _{max}$	$ ho w _{max}$
IA	yes	no	no	0	1	0	0.0216936	9.1340434	0.0040708
IB	yes	no	no	0	0	1	0.0216936	0.0040708	9.1340434
IIA	no	yes	no	1	0	0	9.1340434	0.0216936	0.0040708
IIB	no	yes	no	0	0	1	0.0040708	0.0216936	9.1340434
IIIA	no	no	yes	1	0	· 0	9.1340434	0.0040708	0.0216936
IIIB	no	no	yes	0	1	0	0.0040708	9.1340434	0.0216936

Table 4.1: Plane channel test cases and results

test cases, the grid parameters defined in eqns. (4.1-4.3) are $A_x = A_y = A_z = 1.5$ and $w_x = w_y = w_z = 1.5$. The results are presented in Table 4.1 in maximum absolute value of the velocity components. As expected, alternating the x-direction with y, y with z and z with x we obtain the same pair of numerical errors for the velocities swapping u with v, v with w and w with u.

4.3 Pulse 1D

The following test consists of a density pulse signal moving across an interface with different grid density (see Figure 4.3a). The pulse is defined as:

$$\rho = 1 + \frac{1}{\epsilon e^{\ln(2)\frac{(x-x_0)^2}{2}}}, \qquad (4.12)$$

$$u = 1, \tag{4.13}$$

$$v = w = 0,$$
 (4.14)

where $\varepsilon = 10$ and $x_0 = 4$. The number of points is $N_x \times N_y \times N_z = 47 \times 7 \times 7$ in the streamwise, normal and spanwise directions, respectively, while the physical dimensions are $L_x \times L_y \times L_z = 20 \times 1 \times 1$ with the grid interface at x=4. Without any filter, after a computational of T = 10 time units the flow solution presents some spurious oscillations in correspondence of the sudden coarsening point (see Figure 4.3b). Explicit and implicit sixth order filters (with $\alpha_f = 0$) are applied and compared with the unfiltered solution and among them. The filtered solution presents only a small oscillation around x = 8 and the amplitude of the pulse is better preserved than in the unfiltered solution. The explicit filter gives a result very close to the implicit one just after the interface, but slightly different further downstream. Same results have been found in the work of Visbal and Gaitonde (2001).


(a) Mesh



(b) Comparison between unfiltered solution, implicit and explicit filters

Figure 4.3: Results for pulse 1D test case: comparison between unfiltered solution and response obtained with different filtering schemes.

4.4 Optimization of r.h.s.

For simple geometries, where no stretching or curvilinearity is applied in all the directions, many terms of the r.h.s. subroutine are zero. However, a normal compiler will still consider these terms and the operations between one or more null quantities will require the same computational time of a normal operation. The code can be remarkably speed up if a pre-compiler is able to detect the non zero terms present in the r.h.s. and generate a new file containing the simplified subroutine by omitting those "zero" terms. For example, we consider the following line of the code:

 $Jac(1,1)^{*}(wx(i,j,k,18)^{*}(c43^{*}(wx(i,j,k,8)^{*}Jacodet(1,1) +wx(i,j,k,9)^{*}Jacodet(2,1) +wx(i,j,k,10)^{*}Jacodet(3,1))))$

if the term "Jacodet(1,1)" is equal zero the previous expression become:

Jac(1,1)*(wx(i,j,k,18)*(+wx(i,j,k, 9)*Jacodet(2,1) +wx(i,j,k,10)*Jacodet(3,1)))

Obvious the permanent zero terms are only those linked to the grid transformation, nothing can be pre-decided about the velocity field as it can be zero just for some instants. After the reading of the geometric coordinates, it is possible to evaluate which terms, like "Jac(1,1)", "Jacodet(1,1)", etc., are zero or unitary in the full domain. A cyclic simplification of the different quantities, considering all the mathematical operations, brings to a complete simplification of the r.h.s. subroutine. Figure 4.4 shows a flow-chart illustrating of the pre-compiler program. In order to verify the correctness and the performance of the pre-compiler program, simulations of a laminar flow in a plane channel with different meshing arrangement have been carried out. Four tests cases are considered:

I) fully orthogonal mesh: $\partial x/\partial \xi = \partial y/\partial \eta = \partial z/\partial \zeta \neq 0, 1$ and

 $\partial x/\partial \eta = \partial x/\partial \zeta = \partial y/\partial \xi = \partial y/\partial \zeta = \partial z/\partial \xi = \partial z/\partial \eta = 0;$

II) curvilinear in x-direction mesh: $\partial x/\partial \xi = \partial y/\partial \eta = \partial z/\partial \zeta \neq 0, 1, \partial x/\partial \eta = \partial x/\partial \zeta \neq 0, 1$ and $\partial y/\partial \xi = \partial y/\partial \zeta = \partial z/\partial \xi = \partial z/\partial \eta = 0;$

III) curvilinear in y-direction mesh: $\partial x/\partial \xi = \partial y/\partial \eta = \partial z/\partial \zeta \neq 0, 1, \partial x/\partial \eta = \partial x/\partial \zeta = 0$ $\partial y/\partial \xi = \partial y/\partial \zeta \neq 0, 1 \text{ and } \partial z/\partial \xi = \partial z/\partial \eta = 0;$

IV) curvilinear in z-direction mesh: $\partial x/\partial \xi = \partial y/\partial \eta = \partial z/\partial \zeta \neq 0, 1, \partial x/\partial \eta = \partial x/\partial \zeta = 0$ $\partial y/\partial \xi = \partial y/\partial \zeta = 0$ and $\partial z/\partial \xi = \partial z/\partial \eta \neq 0, 1;$

Table 4.2 presents the results obtained: in the best case (from full 3D to orthogonal) the computational time associated with the r.h.s. subroutine (main cost of the computation) is shortened by a factor of four and a total performance of about a factor of two is achieved. In the other cases the improvement is around 20%, however, this can be a significatively saving of AUs for very large job. Obvious, in all the test cases the correctness of the results has been verified successfully. Finally, the performance also depends on the number of points used in each direction (from here the difference of the results obtained between cases II), III) and IV), which are similar).

Test	time r.h.s.	time r.h.s.	speedup	time step	time step	speedup
case	full 3D	simplified	r.h.s	full 3D	simplified	time step
	[s]	[s]		[s/it]	[s/it]	
I	56.76	14.17	4	0.094	0.049	1.92
II	34.1	28.68	1.19	0.034	0.028	1.21
III	33.29	30.05	1.11	0.033	0.030	1.1
IV	37.13	30.39	1.22	0.037	0.030	1.23

Table 4.2: Performance of the SBLI code with pre-compiler for different test cases: a maximum speed up of a factor ≈ 2 is obtained for fully orthogonal mesh (test *I*).



Figure 4.4: Flow chart of pre-compiler program.

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Chapter 5

Turbulent Flow in a Ridged Channel

In this chapter we are going to use the fully 3D curvilinear code to simulate a 3D turbulent flow of interest. The system is a ridged channel and it presents a stretching in the z-y plane which was possible to simulate with the previous version of the SBLI code only with an inversion of the z axis with the x axis.

Section 5.1 gives an introduction on the turbulent flow in a ridged channel and its practical applications. Section 5.2 gives details on the computational domain and fluid flow conditions; boundary and initial conditions are also given. Section 5.3 presents the results obtained for a laminar flow: this case does not have practical interest, but it is useful in order to check that geometry, mesh and other generic flow conditions are well set up. Section 5.4 presents the results from the turbulent flow simulations carried out at $Re_{\tau} = 150$ and the comparisons with previous experimental and numerical published data (Nezu and Nakagawa (1984), hereafter also indicated as NN; Kawamura and Sumori (1999), KS; Falcomer and Armenio (2002), FA). Differences have been pointed out and discussed. In the successive Section 5.5 the results obtained at an intermediate Reynold number $Re_{\tau} = 360$ are presented. A comparison with the LES of Falcomer and Armenio (2002) performed at $Re_{\tau} = 580$ is presented. Finally, Section 5.6 contain a summary of this chapter.

5.1 Introduction and Previous Works

A ridged channel consists of a channel with longitudinally ridged walls along the streamwise direction of the fluid flow (see Figure 5.1). The fluid is primarily moving in parallel to direction of the ridged walls, but if the Reynolds number is sufficiently high, a pair of large recirculations perpendicular to the main flow can be observed. The intensity of the recirculations can be measured by the streamwise vorticity, while the effect of the ridged walls on the fluid-surface drag coefficient between fluid and surface can be quantified by the the friction velocity along the spanwise direction.

If we consider only half of the channel, this system has practical interest, for example, in the understanding of sediment deposition along rivers. The so called *sand ribbon* effect consists of the sediment of the sand concentrated on the bed of a straight river due to the large recirculation of the water in the spanwise direction. The shape of these sediments can be represented by the longitudinally ridged walls. Due to the negligible effect of gravity on the main and the large recirculation flows, a simulation of a full channel without free surface can be a good approximation of the physical effects presented in a river (see Naot and Rodi, 1982).

Experiments on the ridged channel have been carried out by Nezu and Nakawaga (1984). Here a closed air conduit with ridged walls decided on the bottom and the top of the channel was used to investigate the turbulent structures of the secondary flows. Although the difference with an open channel geometry, from the preliminary water flow tests of Naot and Rodi (1982) it seems that the effect of the free surface is not an essential cause of cellular secondary currents, but it only promotes their intensity near the free surface. Based on this observation, Nezu and Nakawaga measured the

intensity of the secondary recirculation by hot wire anemometer and concluded that the intensity of the recirculation is only 5% of the main streamwise current. They also suggested that the profile of the streamwise velocity along the wall vertical direction can be sufficiently well represented by the log law even above the inclined walls. From this assumption they estimated the wall primary stress, or friction velocity, along the spanwise direction. In the work of Kawamura and Sumori, DNS at low Reynolds number ($Re_{\tau}=150$) has been carried out. Numerically studies have been successively performed by Kawamura and Sumori (1999) and Falcomer and Armenio (2002). In this kind of flow, the failure of a isotropic model is clearly due to the equality of the normal Reynolds stresses. As addressed by Speziale (1984), is the imbalance of these stresses the cause of the large recirculation. Compared with experiments, this value is only one quarter of the actual experimental value ($Re_{\tau} = 580$). However, good agreement has been found, especially in the mean profile and streamwise vorticity values, while major differences are observed in the friction velocity. Falcomer and Armenio performed a LES at same low Reynolds number of Kawamura and Sumori and a LES at same experimental value. The results agree in both cases with the previous published data and additional information have been provided on the coherent structures over the ridged walls. Different from the experiments, in the LES some additional recirculation have been captured close to the ridge corners and the ridge foot. Moreover, the primary wall stress agrees with the experimental one if it is calculated using the same log law suggested by Nezu and Nakawaga.

Similar configuration to the ridged channel flow can be found in Falcomer and Armenio (2004), Hayashi *et al.* (2003) and in Salinas et al. (2004) where the effect of the ridged walls on the heat transfer has been investigated.

In this work, simulations of laminar and turbulent flows have been carried out using the fully 3D curvilinear version of the code. Results have been compared with the previous DNS and LES cited data. Moreover, simulation at an intermediate $Re_{\tau} = 360$ has been carried out in order to better understand the effect of the ridged walls on the turbulence, like the variation of the mean streamwise profile, the reduction of the Reynolds stresses intensity and the effect on the coherent structures.



Figure 5.1: Scheme of a ridged channel and dimensions based on the reference length $\delta = H/2$, where H is the height of the channel.

5.2 Domain, Grid and Computational Conditions

Figures 5.2a and 5.2b shows the computational domain and a typical grid for the ridged channel, respectively. Choosing as reference length half height of the channel $(\delta = H/2)$, the dimensions of the domain for the different test cases are shown in Table 5.1 (test cases A and B are referred to laminar flow, while I, II and III to turbulent flow). In the experiments of Nezu and Nakagawa (1984), a wider domain has been used (4.5 δ) due to the confinement of the fluid. A main requirement for the correct simulation of a fluid flow system is to have a domain large enough to contain the maximum turbulent flow structure. In a plane channel flow this is around three times the height of the channel ($\delta\delta$). However, the dimensions chosen here (1.25 $\pi\delta$) are similar to those of Kawamura and Sumori (1999) which satisfy the "minimal flow unit" conditions suggested by Jiménez (2006) for a plane channel flow. To confirm,

a two point correlation analysis has been carried out. For the streamwise direction, the two point normalized correlation is defined as:

$$R_{11} = \frac{u'(x_0 + r, y_0, z_0)u'(x_0, y_0, z_0)}{(u'(x_0, y_0, z_0))^2},$$
(5.1)

where u' is the fluctuation of the velocity. The overbar indicates the time average and r is the distance from the reference point of coordinates x_0 , y_0 , z_0 (here chosen at the inlet of the domain). A similar definition is used for the correlations in the vertical and spanwise directions, R_{22} and R_{33} , respectively.



Figure 5.2: Scheme and grid of ridged channel.

Figures 5.3 and 5.4 show the two point correlation for the test case I in the streamwise and the spanwise directions at two different vertical locations (the centerline and close to the wall, $y^+ = 19$). Along the streamwise direction, whatever is the distance from the wall, half domain is wide enough for the decay of vertical and streamwise velocity correlations. Differently, the streamwise component needs a longer domain. Similar situation is found in the spanwise direction: all the correlations decay nearly to zero in half spanwise length, apart from the streamwise velocity at the centerline which achieves only the 50% of the decay. Similar discrepancies are also found in the work of Li (2003), where similar computational dimensions have been used ($L_x = 3\delta$, $L_y = 2\delta$, $L_z = 1.5\delta$). A possible explanation seems to be the presence of the "acoustic resonance" effect, as explained in the work of Coleman *et al.* (1995). However, in both directions, the results at $y^+=19$ are better than centerline values. This is quite obvious if we consider that at center channel the flow structures are bigger than close to wall.

Test Case	$Re_{ au}$	L_{x}	L_y	L_z	r_1	r_2	r_3
A	2	$1.25\pi\delta$	2δ	$0.375\pi\delta$	0.26δ	0.52δ	0.125δ
В	2	3.12δ	2δ	2.6δ	0.78δ	1.04δ	0.125δ
I & II	150	$1.25\pi\delta$	2δ	$0.375\pi\delta$	0.26δ	0.52δ	0.125δ
III	360	$1.25\pi\delta$	2δ	$0.375\pi\delta$	0.26δ	0.52δ	0.125δ

Table 5.1: Dimensions for ridged channel simulations.



(a) Correlation at centerline

(b) Correlation at $y^+ = 19$

Figure 5.3: Two point correlation for test case I along x-direction at different y^+ locations.



(a) Correlation at centerline (b) Correlation at $y^+ = 19$

Figure 5.4: Two point correlation for test case I along z-direction at different y^+ locations.

The mesh is uniform in the streamwise and spanwise directions, while a hyperbolic tangent stretching is applied along the vertical direction. Table 5.2 gives the computational parameters of the different tests and the grid resolution. Test II will be the reference test, hereafter indicated as *present DNS*.

Test Case	Re_{τ}	N_x	N_y	N_z	N _{tot}	$\triangle x^+$	$\bigtriangleup y^+$	$\triangle z^+$
Α	2	7	49	37	12691	1.31	0.02-0.12	0.065
В	2	7	49	61	20923	1.04	0.02-0.12	0.09
Ι	150	13	49	37	23569	49.06	1.73-10.34	4.91
II	150	49	49	37	88837	12.27	1.73-10.34	4.91
III	360	113	129	70	1020390	12.62	0.74-11.67	6.14

Table 5.2: Computational parameters for ridged channel simulations.

The Reynolds number Re_{τ} is based on the friction velocity defined as $u_{\tau p}$ =

 $\sqrt{\frac{\delta}{\rho_b}} \left(-\frac{\partial p}{\partial x}\right)$, on the wall temperature T_w (and then the wall viscosity ν_w), the bulk density ρ_b , defined in section 2.2, and the above chosen reference length δ :

$$Re_{\tau} = \frac{\rho_b u_{\tau p} \delta}{\nu_w}.$$
(5.2)

The usual friction velocity $u_{\tau} = \sqrt{\frac{\tau_w}{\rho_b}} = \sqrt{\frac{\nu_w}{\rho_b} \frac{\partial \tilde{u}}{\partial y}} \Big|_w$ could not be used in this case due to the presence of the ridge. However, an averaged value along the spanwise direction can be used $\langle u_{\tau} \rangle = \sqrt{\frac{\langle \tau_w \rangle}{\rho_b}} = \sqrt{\frac{\nu_w}{\rho_b} \frac{\partial \langle u \rangle}{\partial y}} \Big|_w$ instead of it, where the $\langle \rangle$ indicates the averaged value in the spanwise direction. The difference between Re_{τ} and the Reynolds number based on $\langle u_{\tau} \rangle$ (hereafter indicated as $Re_{\langle \tau \rangle}$) is quite small. For example, at $Re_{\tau} = 150$ correspond a value of $Re_{\langle \tau \rangle} \approx 140$.

Periodic conditions are applied in the x and z directions, while no-slip conditions are applied at the top and bottom walls. Moreover, the flow is driven by a constant pressure gradient $\partial p/\partial x = -1$.

The turbulent flow simulations have been carried out at a Mach numbers, based on the friction velocity $u_{\tau p}$, of M = 0.1. The corresponding centerline Mach number is ≈ 1.8 and the hypothesis of incompressibility need to be verified a posteriori. Lower values of the Mach number require a smaller time step (which increases the computational effort) and, then, they have been not considered.

In the laminar flow simulation the entropy splitting has been switched off by setting the entropy factor $\beta = 4 \times 10^{10}$, while, in the turbulent flow simulation, a good stability has been provided by setting $\beta = 4$ following the work of a plane channel DNS (see Li, 2003).

For the turbulent flow, a mean turbulent profile, obtained by the law of the wall and the log law, has been used as initial conditions. Moreover, in order to artificially introduce the turbulent flow, some perturbations are added on to the mean profile (see Li, 2003). The perturbations need to be correlated and here are described by the following equations:

v'

$$u' = AL_x \sum \cos\left((2N-1)\beta_x x + \frac{N-1}{2}\right) \sin\left((2N-1)\pi y\right) \\ \times \sin\left((2N-1)\beta_z z + 0.4(N-1)\right), \qquad N = 1,3$$
(5.3)

$$= -A \sum \sin\left((2N-1)\beta_x x + \frac{N-1}{2}\right) (1 + \cos\left((2N-1)\pi y\right)) \\\times \sin\left((2N-1)\beta_z z + 0.4(N-1)\right), \qquad N = 1,3$$
(5.4)

$$w' = -\frac{A}{2}L_z \sum \sin\left((2N-1)\beta_x x + \frac{N-1}{2}\right) \sin\left((2N-1)\pi y\right) \\ \times \cos\left((2N-1)\beta_z z + 0.4(N-1)\right), \qquad N = 1,3$$
(5.5)

where $\beta_x = 2\pi/L_x$ and $\beta_z = 2\pi/L_z$. The value of the coefficient A is chosen equal to 10% of the centerline velocity in order to guarantee a convergence to fully developed turbulent flow after $\approx 15\delta/u_{\tau p}$ time units, as it occurs for the plane channel case presented in the work of Li (2003). As in the work of Kawamura and Sumori (1999), the results are presented in four chosen locations: ridge center $(z/\delta = 0, i)$; ridge corner $(z/\delta = 0.13, ii)$; ridge foot $(z/\delta = 0.26, iii)$; trough center $(z/\delta = 0.6, iv)$.

The computational costs for the simulations here presented are summarized in Table 5.3.

Test Case	N _{tot}	$Procs_x$	$Procs_y$	Procs _z	$Procs_{tot}$	time [h]	Cost [AUs]
A	12691	1	8	4	32	0.2	29
В	20923	1	8	4	32	2	302
I	23569	1	8	4	32	6	873
II	88837	4	4	2	32	18	2799
III	1020390	4	4	2	32	124	19074

Table 5.3: Computational costs for ridged channel simulations

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5.3 Laminar Flow Results

We present in this section the results for a laminar flow in a ridged channel. This has been carried out in order to check that geometry, mesh and other generic flow conditions are well set up and also in order to have a qualitative idea of the fluid flow behavior. A profile similar to the analytical solution for steady state plane channel is expected in each z - y cross section. However, due to the geometry of the system, there are some small differences. Figure 5.5 shows a typical contour plot of streamwise laminar velocity for a ridged channel. If we apply the N-S equations to an infinitesimal element in the bulk of the system we find that, mathematically, the only difference with the plane channel is in the non-zero second derivative of streamwise velocity $\partial^2 u/\partial z^2$. The streamwise momentum equation is:

$$0 = -\frac{\partial p}{\partial x} - \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} - \frac{1}{Re} \frac{\partial^2 u}{\partial z^2},$$
(5.6)

(note the presence of partial derivatives instead of total derivatives). We expect,



Figure 5.5: Typical contour levels line in a laminar channel flow.

then, that far from the ridge the flow converges to the plane channel solution. This is confirmed by test case B, where, as shown in tables 5.1, a wider domain has been used. The simulations have been carried out at Re = 2 and M = 0.1 and a zero velocity field has been used as initial condition.

In Figures 5.6 (a-b) are shown the steady state results obtained after 10 time units (the convergence of the instantaneous flow has been checked and achieved) and compared with the theoretical profile for plane channel on the position (i) and (iv) previously defined. As expected, the solution for a larger domain show a smaller effect of the ridge on the flow. Finally, Figures 5.7 (a-b) show the non zero value of the second derivative of the streamwise velocity $(\partial^2 u/\partial z^2)$ along the vertical direction for both test cases.



(b) Comparison at $z/\delta = 0.6$ (iv)

Figure 5.6: Streamwise velocity in a laminar ridged channel case at different spanwise locations.



(a) Comparison at $z/\delta = 0$ (i) (b) Comparison at $z/\delta = 0.6$ (iv)

Figure 5.7: Second derivative of streamwise velocity $\partial^2 u/\partial z^2$ at different spanwise locations.

5.4 Turbulent Flow Results at $Re_{\tau} = 150$

The statistics data have been collected after $40\delta/u_{\tau p}$ time units and for successively $160\delta/u_{\tau p}$ time units until statistical convergence is achieved. Moreover, considering the homogeneity and periodicity along the streamwise direction, data have been also averaged in this direction. The results are presented in six subsections: mean streamwise velocity, large recirculation, vorticity and friction velocity, turbulent intensity, coherent structures and budget of the mean streamwise momentum.

5.4.1 Mean Streamwise Velocity

All the quantities are averaged using the Favre averaged (or density-weighted average) definition. The contour plot of the streamwise velocity \tilde{u} normalized by \tilde{u}_{max} is represented in Figure 5.8a, while Figure 5.8b shows a comparison with Kawamura and Sumori (1999) data. Results from two simulations are very close: the shape of the contours follows the shape of the ridge with a progressive tendency to plane channel flow on the middle of the ridge and on the extremes of the domain. However, at position (i) the large recirculations interact with each other and their global effect is to pull up the fluid flow. Same thing occurs at location (iv), where the effect is to push the fluid towards the wall. On the middle of the ridge, at $y/\delta \simeq 7$, the fluid has achieved the 95% of the maximum velocity, while, on the trough center the same intensity is achieved at $y/\delta \simeq 6$. In both cases a deviation from the log wall is obtained and can be better observed in Figure 5.9, where $u^+ = \tilde{u}/u_\tau$ versus $y^+ = (y - y_{wall}) u_{\tau p} / \nu_w$ is plotted, where $y - y_{wall}$ is the vertical distance from the wall surface. If we consider the values at the trough center, where the flow is more similar to a plane channel flow and, then, to the law of the wall and log wall, the Mach number does not seem to have an effect on the mean profile. This is in agree with our hypothesis of incompressibility discussed in the section 5.2.





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Figure 5.8: Contour plot of mean streamwise velocity and comparison with the numerical work of Kawamura and Sumori (1999).



Figure 5.9: Mean streamwise velocity: deviation from theoretical profile (law of the wall and log law) at $Re_{\tau} = 150$.

5.4.2 Large Recirculation

A vector plot of vertical and spanwise velocities is given in Figure 5.10. The center of the recirculation is about $z/\delta = \pm 0.27$ and $y/\delta = 0.31$, very close to previous works ($z/\delta = \pm 0.25$ and $y/\delta = 0.33$ in Kawamura and Sumori (1999)). Moreover, in the middle of the ridge and on the extremes of the domain the large recirculation is still presents. Similar behavior is observed in Kawamura and Sumori (1999) and Falcomer and Armenio (2002) results shown in Figure 5.11. In Figures 5.12a and 5.12b are plotted the intensity of the vertical and spanwise velocities along the vertical coordinate at different spanwise locations for present DNS.



Figure 5.10: Large recirculation for present DNS represented by the vector plot of vertical and spanwise mean velocities.



(a) Kawamura and Sumori (1999) results



(b) Falcomer and Armenio (2002) results

Figure 5.11: Large recirculation of previous numerical works visualized by the vector plot and streamline of vertical and spanwise velocity.



(b) Spanwise velocity

Figure 5.12: Vertical and spanwise mean velocities in the ridged channel along the vertical direction and at different spanwise locations.

5.4.3 Vorticity and Friction Velocity

Two important parameters for the ridged channel flow are the streamwise vorticity $(\tilde{\omega}_x = \partial \tilde{w}/\partial y - \partial \tilde{v}/\partial z)$ and the friction velocity u_{τ} along the spanwise direction. Contour lines of normalized vorticity $\tilde{\omega}_x \delta/\tilde{u}_{max}$ are shown in Figure 5.13. At the center of the recirculation the value of $\tilde{\omega}_x \delta/\tilde{u}_{max} = 0.34$. Compared with the experiments of Nezu and Nakagawa (1984) (see Figure 5.15, case I) the vorticity values here found are overestimated of about 70%, but are qualitatively similar. However, the intensity value found are similar to those of Kawamura and Sumori (1999) and Falcomer and Armenio (2002) results (see Figures 5.14a and 5.14b). The discrepancies can be due to an experimental measurement considering the low intensity of the vertical and spanwise velocities.



Figure 5.13: Contour plot of normalized streamwise vorticity $(\tilde{\omega}_x = \partial \tilde{w}/\partial y - \partial \tilde{v}/\partial z)$ for half ridged channel in the present DNS. The negative regions which appear near the wall are not depicted for clear representation of the streamwise vorticity.

The friction velocity u_{τ} , normalized by the $\langle u_{\tau} \rangle$ averaged friction velocity defined in

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Figure 5.14: Contour plot of normalized streamwise vorticity obtained in the work of Kawamura and Sumori (1999).

section 5.2, is plotted in Figure 5.16. In the same figure are shown the value obtained in the previous numerical works: the shape of the friction velocity is similar but the maximum and minimum value in the location i, ii, iii, iv are slightly different (the maximum peak in the ridge corners is around the 13% higher than in Kawamura and Sumori (1999) and 6% higher than Falcomer and Armenio (2002) results. On the extreme of the domain the present DNS is around the 4.5% higher than that obtained by Kawamura and Sumori (1999) and Falcomer and Armenio (2002)).



Figure 5.15: Contour plot of normalized streamwise vorticity obtained in the experimental work of Nezu and Nakagawa (1984).



Figure 5.16: Friction velocity for present DNS along the spanwise direction. The values are compared with the work of Kawamura and Sumori (1999) and Falcomer and Armenio (2002).

5.4.4 Turbulent Intensity Quantities

Figures 5.17 (a-d) show the u_{rms} values normalized by the maximum mean streamwise velocity \tilde{u}_{max} for the present DNS, Kawamura and Sumori (1999) and a turbulent plane channel flow simulation obtained at the Reynolds number, based on the friction velocity u_{τ} , of 150. The dimensions of the plane channel are similar to those of the ridged channel (see Kawamura and Sumori, 1999). As usually, the values are plotted at the four spanwise location (*i*, *ii*, *iii* and *iv*). The comparison presents some discrepancies: in the present DNS the values are overestimated and the effect of the ridge, compared to the plane channel, seems to reduce the turbulent intensity by only 10% - 30%, while in Kawamura and Sumori (1999) the reduction range is about 20% - 50%. Moreover, in the bulk of the channel the values are slightly higher (0.05 and 0.04, respectively for the present DNS and Kawamura and Sumori (1999)).

For the v_{rms} and w_{rms} values, as shown in Figures 5.18 (a-d) and 5.19 (a-d) respectively, good agreement is found, except for the bulk region where a small difference is noted.

Finally, Figures 5.20 (a-b) show the Reynolds shear stresses -u'v' and -u'w' for present DNS. The value found here are nearly twice larger than that of Kawamura and Sumori (1999) (Figures 5.20, c-d), but qualitatively similar.



(c) Comparison at z = 0.26 (iii)

(d) Comparison at z = 0.6 (iv)

Figure 5.17: Turbulence intensities values u_{rms} normalized by the maximum streamwise velocity \tilde{u}_{max} at different spanwise locations for present DNS. The data are compared with plane channel results obtained at $Re_{\tau} = 140$.





Figure 5.18: Turbulence intensities values v_{rms}/\tilde{u}_{max} at different spanwise locations for present DNS.





Figure 5.19: Turbulence intensities values w_{rms}/\tilde{u}_{max} at different spanwise locations for present DNS.



Figure 5.20: Reynolds shear stresses at different spanwise locations for present DNS and for the numerical work of Kawamura and Sumori (1999).

5.4.5 Coherent Structures

A good method to visualize the coherent structures presented in a turbulent flow is the λ_2 criteria of Jeong and Hussain (1995). It is based on the second eigenvalue λ_2 of the sum of the square of the symmetric tensor $\overline{S_{ij}}$ and anti-symmetric tensor $\overline{\Omega_{ij}} = (1/2)(\partial \tilde{u}_i/\partial x_j - \partial \tilde{u}_j/\partial x_i)$ components of the gradient of the velocity field. Figure 5.21 shows the coherent structures for the ridged channel at $\lambda_2 = 100$ and colored by the pressure field. The structures are similar to those found in a plane channel flow at same Reynolds number, however Falcomer and Armenio pointed out that the effect of the ridged seems to be an alignment of the structures along the longitudinally walls. The secondary recirculation does not seem to have an effect on the coherent structures. In Figure 5.22 the structures appear quite long and thick and partially tilted as seen in a turbulent plane channel. The length of a single structure is around 400 wall units, while the total length of a chain constituted overlapping more structures is around 1000 wall units. Figure 5.23 shows the structure in the x - y plane.



Figure 5.21: Coherent structures in the ridged channel for present DNS obtained with the criteria of second eigenvalue λ_2 of the sum of the square of the symmetric tensor $\overline{S_{ij}}$ and anti-symmetric tensor $\overline{\Omega_{ij}} = (1/2)(\partial \tilde{u}_i/\partial x_j - \partial \tilde{u}_j/\partial x_i)$ components of the gradient of the velocity field. The structures are shown for the values of $\lambda_2 = 100$.



Figure 5.22: Coherent structures at $\lambda_2 = 100$ in the x - z plane. The effect of the ridged seems to consists in an alignment of the structures along the ridged itself.



Figure 5.23: Coherent structures at $\lambda_2 = 100$ in the x - y plane.

5.4.6 Budget of Mean Streamwise Momentum Equation

For a fully developed turbulent flow, individual terms in the Reynolds averaged momentum equation in each direction has to be balanced. The streamwise mean momentum equation for a fully developed turbulent channel flow is:

$$0 = \underbrace{-\frac{\partial \overline{p}}{\partial x}}_{\text{Pre}} + \underbrace{\frac{\partial \left(\overline{p}\widetilde{u}\widetilde{v}\right)}{\partial y} + \frac{\partial \left(\overline{p}\widetilde{u}\widetilde{w}\right)}{\partial z}}_{\text{Conv}} + \underbrace{\frac{1}{Re}\left(\frac{\partial \overline{\tau_{xy}}}{\partial y} + \frac{\partial \overline{\tau_{xz}}}{\partial z}\right)}_{\text{Visc}} + \underbrace{\left[-\frac{\partial \left(\overline{p}\widetilde{u''v''}\right)}{\partial y} - \frac{\partial \left(\overline{p}\widetilde{u''w''}\right)}{\partial z}\right]}_{\text{T1+T2}}.$$
(5.7)

The first, second, third and final term of eqn. (5.7) are the pressure gradient, the convective term, the viscous term and the Reynolds shear stresses variation, respectively. Positive quantities accelerate the fluid (gain), while negative quantities decelerate it (loss). For a steady turbulent channel the budget of the averaged equation has to be zero. In Figures 5.24a-b and 5.25a-b are shown the budgets for the streamwise momentum of the ridged channel along the vertical direction.

In the middle of the ridge (i), near the wall the fluid is accelerated by the pressure gradient term and the two turbulent quantities, while the convection term decelerates it. At the trough center (iv), the convection and the turbulent terms are balanced by the viscosity term that decelerates the fluid. In both case, far from the wall, the pressure gradient is balanced by the Reynolds shear stress T1, which, as already found by Kawamura and Sumori (1999), looks to play a fundamental role in the mean flow field. The absolute maximum imbalance found is around 7% of the biggest term. At the ridge corner (ii) and ridge foot (iii) the budgets are not satisfactory: the imbalance are, respectively, around the 60% and 25% of the maximum values. The cause can be the lack of resolution in the vertical direction considering that only 5 points are used within the sublayer region.



Figure 5.24: Budget of the streamwise momentum equation at location i and ii. The graph shows also the convective, pressure, viscous and turbulent stress terms.



Figure 5.25: Budget of the streamwise momentum equation at location iii and iv. The graph shows also the convective, pressure, viscous and turbulent stress terms.

5.5 Turbulent Flow Results at $Re_{\tau} = 360$

Measurements of Nezu and Nakagawa have been numerically carried out at $Re_{\tau} = 580$. A direct numerical simulation at this Reynolds number is computationally too expensive; however, in order to better understand the effect of the ridge on the turbulence, a simulation at an intermediate value of $Re_{\tau} = 360$ has been performed.

Table 5.2 shows the grid parameters used here. Grid and scheme are similar to those shown in Figure 5.2a and Figure 5.2b. As in the low Reynolds number case, the grid is uniform in x and z directions and a tangent hyperbolic stretching in the vertical direction is applied. The grid resolution in the streamwise and spanwise direction are $\Delta x^+ = 12.62$ and $\Delta z^+ = 6.14$, respectively and the first point in the vertical direction is about $y^+ = 0.74$ with a total of 10 points within the viscous sublayer.

The two point correlations at the centerline are shown in Figure 5.26a and 5.26b: the streamwise velocity correlation is around 0.4 on half a domain (as in the low Reynolds case), while the vertical and spanwise velocities are approaching to zero. Figure 5.27a and Figure 5.27b show the two point correlation close to the wall at a distance of $y^+ = 45$ over the ridge: all the values approach to zero.



(a) Streamwise direction

(b) Spanwise direction

Figure 5.26: The two point correlation along streamwise and spanwise direction for $Re_{\tau} = 360$ test case at the centerline.



(a) Streamwise direction

(b) Spanwise direction

Figure 5.27: Two point correlation along streamwise and spanwise direction $Re_{\tau} = 360$ test case at $y^+ = 45$.
5.5.1 Mean Streamwise, Vorticity and Friction Velocity

Figure 5.28a shows the contours of the mean streamwise velocity \tilde{u} normalized by the \tilde{u}_{max} . The lines on the middle of the ridge have a stronger deflection than the corresponding low Reynolds case (see Figure 5.28b), i.e. a more intensive recirculation is present and the pull-up effect on the streamwise velocity is stronger. However, the mean streamwise velocity along the vertical direction seems now to be closer to the law of the wall and the log wall (see Figure 5.29). This means that the effect of the ridge is less influent at higher Reynolds number. Also in this case, the Mach number does not seem to have an effect on the mean profile.

Figure 5.30a and Figure 5.30b show the vector and streamline plot of the vertical and spanwise velocities. The center of the recirculation is close to the experimental results of Nezu and Nakagawa (1984): $z/\delta = \pm 0.24$ and $y/\delta = 0.28$. At the foot of the ridge it is possible to observe a small recirculation which was not captured in the experimental data. The intensity of the vertical and spanwise velocities in this additional recirculation seems to be very small if compared to the large recirculation (see Figures 5.32a and 5.32b). In the experimental results, this recirculation has not been captured probably due to the absence of measurement points in that location or the low intensity of the velocities. This can explain the failure of the experimental test into capture it. Similar additional recirculations have been captured by Falcomer and Armenio (2002) in their LES: one in the same position of the present DNS and another close to the upper corner of the ridge (see Figure 5.31).

The maximum intensity of the primary large recirculation is $\tilde{\omega}_x \delta/\tilde{u}_{max} = 0.49$, while for the small one the maximum intensity is $\tilde{\omega}_x \delta/\tilde{u}_{max} = 0.30$ at $z/\delta = \pm 0.32$ and $y/\delta = 0.049$.

Finally, Figure 5.33 shows the friction velocity compared with the low Re case. The maximum values and the values at the extremes of the domain are close to each other, but at the ridge foot the minimum value, at $Re_{\tau} = 360$, is about 20% lower.





Figure 5.28: Contour plot of mean streamwise velocity normalized by the maximum value and compared with the $Re_{\tau} = 150$ case.



Figure 5.29: Deviation from theoretical profile (law of the wall and log law) at $Re_{\tau} = 360$. The value are presented at different spanwise locations.



(a) Vector plot

(b) Streamline plot

Figure 5.30: Contour plot and streamline of vertical and spanwise velocity showing the large recirculation at $Re_{\tau} = 360$. A small additional recirculation is captured at the foot ridge corner.



Figure 5.31: Streamline of vertical and spanwise velocity in the work of Falcomer and Armenio (2002) at $Re_{\tau} = 580$. The figure show the large recirculation and two additional small recirculations are captured at the upper and foot ridge corner.



(a) Vertical velocity

(b) Spanwise velocity

Figure 5.32: Mean vertical and spanwise velocities in the ridged channel along the y-coordinates and at $Re_{\tau} = 360$. The values are normalized by the maximum streamwise velocity \tilde{u}_{max} and are presented at different spanwise locations.



Figure 5.33: Friction velocity in the ridged channel along the spanwise direction and at $Re_{\tau} = 360$. The values are normalized by the average friction velocity $\langle u_{\tau} \rangle$ and compared with the $Re_{\tau} = 150$ case.

5.5.2 Turbulent Intensities, Coherent Structures and Budget

In Figures 5.34, 5.35 and 5.36 are shown the root mean square values normalized by \tilde{u}_{max} at the four spanwise locations previously defined. The results are similar to plane channel equivalent to $Re_{\tau} = 150$ ridged channel, here indicated by the dot symbol. This means that at higher Reynolds number the effect of the ridged walls is less influent than at low Reynolds numbers. However, some differences are found: in the ridged channel the pick value of the u_{rms} looks shifted towards the wall. A lighter similar effect is noted in the w_{rms} values but not in the v_{rms} . Moreover, the u_{rms} , at the ridge foot, is reduced by 20%. A smaller reduction is observed in the vertical and spanwise turbulence intensities. Figure 5.37 gives the Reynolds shear stresses $\overline{-u'v'}$

In Figure 5.38 are shown the coherent structures for fully developed flow ($t = 120\delta/u_{\tau}$) at $\lambda_2 = 700$ and colored by the pressure field. The structures look aligned to the ridged walls, as better shown in Figure 5.39. The averaged length of each structure is around 250 wall unit, while a chain obtained overlapping the structures is around 800 wall unit. These values agree the corresponding data from Falcomer and Armenio (2002). From Figure 5.40 it is possible to observe the inclination of the structures, similar to those found in a plane channel (see Jeong *et al.*, 1997). Also in this case, the secondary recirculation does not seem to have an effect on the coherent structures.

Finally, Figure 5.41 and Figure 5.42 show the budget balance for the streamwise momentum equation. Similar to the low Reynolds case, the right hand side quantities are collected in four terms: pressure gradient $-\frac{\partial \bar{p}}{\partial x}$, convective $\frac{\partial(\bar{p}\tilde{u}\tilde{v})}{\partial y} + \frac{\partial(\bar{p}\tilde{u}\tilde{v})}{\partial z}$, viscous term $\frac{1}{Re} \left(\frac{\partial \bar{\tau}_{xy}}{\partial y} + \frac{\partial \bar{\tau}_{xz}}{\partial z} \right)$ and Reynolds shear stress variation $-\frac{\partial(\bar{p}\tilde{u}\tilde{v}')}{\partial y} - \frac{\partial(\bar{p}\tilde{u}\tilde{v}')}{\partial z}$. On the ridge center (i) the viscous term is balanced by the variation of the first Reynolds shear stress T1, while the convective terms are balanced by the second Reynolds shear stress T2. Same situations occurs at the extreme of the domain (iv), but the convective and T2 quantities are here much smaller. As expected, on the ridge corner (ii) and ridge foot (iii) the budget is better than low Reynolds case due to the higher resolution used (here 10 points within the sublayer region, while only 5 points in the low Reynolds case). The imbalance is only around the 10% of the maximum quantities. It is interesting to note that while on the corner ridge (ii) the convective term decelerates the flow, at the corner foot the convection accelerates it. Same behaviour is noted in the T2 term.



Figure 5.34: Turbulence intensities for the streamwise velocity at $Re_{\tau} = 360$ and normalized by the maximum mean streamwise value. The data are compared with the plane channel case at $Re_{\tau} = 150$.



Figure 5.35: Turbulence intensities for the vertical velocity at $Re_{\tau} = 360$ and normalized by the maximum mean streamwise value. The data are compared with the plane channel case at $Re_{\tau} = 150$.



Figure 5.36: Turbulence intensities for the spanwise velocity at $Re_{\tau} = 360$ and normalized by the maximum mean streamwise value. The data are compared with the plane channel case at $Re_{\tau} = 150$.



(a) $\overline{-u'v'}$



Figure 5.37: Reynolds shear stresses for the ridged channel at $Re_{\tau} = 360$.



Figure 5.38: Coherent structures at $Re_{\tau} = 360$ visualized using the criteria of second eigenvalue λ_2 of the sum of the square of the symmetric tensor $\overline{S_{ij}}$ and anti-symmetric tensor $\overline{\Omega_{ij}} = (1/2)(\partial \tilde{u}_i/\partial x_j - \partial \tilde{u}_j/\partial x_i)$ components of the gradient of the velocity field. The structures are shown for the value of $\lambda_2 = 800$ and colored using the pressure values.



Figure 5.39: Coherent structures with $\lambda_2 = 800$ in the x - z plane and colored using the pressure values.



Figure 5.40: Coherent structures with $\lambda_2 = 800$ in the x - y plane and colored using the pressure values.



Figure 5.41: Budget balance for the ridged channel at $Re_{\tau} = 360$ and at the center ridge and trough channel.



Figure 5.42: Budget balance for the ridged channel at $Re_{\tau} = 360$ and at the upper and foot ridge corner.

5.6 Summary

In this Chapter we presented the simulation of a turbulent flow in a ridged channel at two different Reynolds numbers, 150 and 360. At Reynolds number $Re_{\tau} = 150$ results agree well with previous numerical and experimental works. Some differences in the friction velocity and Reynolds stresses are found. Budgets analysis is satisfactory at the extreme of the domain (iv) and on the middle channel (i), but not at the ridge foot (iii) and upper corner (ii) probably due to a lack of resolution in that region. At Reynolds number $Re_{\tau} = 360$ the flow behavior is qualitatively in good agreement with the experimental data provided by Nezu and Nakagawa (1984) and the LES of Falcomer and Armenio (2002). A small additional recirculation is captured at the ridge foot similar to those observed by Falcomer and Armenio. Budget analysis is satisfactory in location (i) and (iv) and nearly satisfactory in (ii) and (iii). A complete set of data are now available at low and intermediate Reynolds numbers. Finally, the kind of geometry presented in this chapter is stretched only in a bidimensional plane and more general cases need to be considered in order to further validate the fully 3D version of the code. In the next Chapter a validation study on a more generic three-dimensional shape will be presented.

Chapter 6

Turbulent Flow Over an Axisymmetric Hill

We present in this chapter the main simulation of the thesis work. The goals are to further validate the new version of the SBLI code and to study the 3D boundary layer separation mechanism which occurs behind an axisymmetric hill. An introduction to previous experimental and numerical studies is given in Section 6.1, while details on the geometry, grid resolution and boundary conditions are given in Section 6.2. A complete Section 6.3 is dedicated to the inflow condition generation and implementation in particular. The results are presented in two parts: Section 6.4 gives the results from a coarse grid simulation, while Section 6.5 contains the results obtained from a fine grid simulation. Details on the mean velocity profiles, turbulence quantities, coherent structures and laminarisation issues are analyzed and discussed. Finally, Section 6.6 gives the summary.

6.1 Introduction

Previous experimental investigation of a boundary layer flow over an axisymmetric hill has been carried out by Ishihara et al. (1999). The obstacle had a cosine-squared cross section and the ratio between the approaching boundary layer thickness δ and the hill height H was 9 and the Reynolds number 1.1×10^4 , based on the free stream quantities. Although little has been presented on the flow separation and the near wall turbulence structures, it was clear that, while the flow accelerated over the top and around the side of the hill, flow separation and reattachment occurred on the leeside of the hill. A further study has been presented by Simpson et al. (2002) where the mean surface pressure, oil-flow visualizations and three-velocity-component laser-Doppler velocimeter measurements were presented. The ratio between the incoming turbulent boundary layer thickness and the hill height $(\delta/H = 1/2)$ was much smaller than that of Ishihara et al. (1999). In Simpson experiment, complex vortical flow separations occurred on the leeside of the hill and merged into two larger streamwise vortices downstream. Precise measurements of the mean velocities, the turbulence Reynolds stresses, the triple products and the skin friction in the near wall region were presented. In particular, the flow topology obtained by the oil-flow visualization suggested the presence of multiple flow separation and re-attachment points occurring over a large area on the leeside of the hill.

The first numerical study of the Simpson hill flow has been performed by Patel *et al.* (2003) using a large eddy simulation technique. The mean surface pressure, flow visualization and mean velocity profiles were presented and compared with the experiments of Simpson *et al.* (2002). The agreement was generally good, but some differences were observed in the flow topology detected over the hill. In another study of Wang *et al.* (2004), five turbulence models were used: the Craft-Launder-Suga cubic eddy-viscosity model (Craft *et al.*, 1996); the Apsley and Leschziner (1998) cubic eddy-viscosity model (AL- ϵ); the Wallin and Johansson (2000) explicit algebraic stress

model (WJ- ω); the Abe-Jang-Leschziner quadratic eddy-viscosity model (2003)(AJL- ω); and the Speziale-Sarkar-Gatski Reynolds-stress-transport model (Speziale et al., 1991). Both a periodic 2D hill and a single 3D hills have been investigated in order to examine whether the predicted performance in three-dimensional conditions related to that in two-dimensional case. It was found that 2D hill geometry different separation behavior was predicted by different turbulent models and, for the 3D hill, none of them was able to capture the flow topology observed by Simpson et al (2002). In fact, only a single vortex pair associated with a single separation line on the leeward side of the hill were detected. Similar conclusions were made in in the work of Temmerman et al. (2004) where a second-moment-closure RANS modeling was presented. In a recent experimental work of Byun et al. (2004), measurement results for two different hill heights $H=\delta$ (small bump) and $H=2\delta$ (large bump) were presented. While the flow topology for the large bump was still quite similar to that suggested by Simpson et al. (2002) a single vortex pair was found for the small bump configuration, in consistency to previous numerical investigations. Further studies of Byun et al. (2006) using a three-dimensional fiber-optic Laser Doppler Velocimeter (LDV), have shown a similar flow topology to that predicted by numerical simulation, even for the $H=2\delta$ case (large bump), indicating the difference with previous experimental work being attributed to the effect of gravity on the oil-flow mixture. Numerical investigations using RANS, LES and detached eddy simulation (DES) have been carried out by Persson et al. (2006). The results have shown that while the RANS fails to predict several important flow features, both LES and DES are clearly capable of reproducing the correct flow separation pattern. However, to produce the correct predictions, the near-wall grid resolution must be increased substantially and in particular in the spanwise direction. Moreover, they found that the pressure field is sensitive to the location of the inlet and the DES model is even more sensitive to the inlet boundary condition on the eddy viscosity profile. A large eddy simulation combined with a zonal near-wall approximation has been presented by Tessicini et al. (2007). In their zonal scheme, the state of the near-wall layer of the flow is described by the parabolized Navier-Stokes equations solved on a sub-grid embedded within a global LES mesh. The wall shear stress found from the solution of the boundary layer equations is used in the LES domain as a wall boundary condition. Satisfactory results (same flow topology and mean quantities of experimental data) are achieved using a very fine grid (up to 9.6 million points), while the simulation completely fails to capture the separation process on a coarse grid (1.5 million points). A visualization of the structures of flow separation behind the hill was given in Patel et al. (2007) by LES. Streamwise, wall normal and spanwise vorticity iso-surfaces in combination with pressure gradient iso-surfaces were used to visualize the flow structure and they concluded that the separation and re-attachment processes are strongly controlled by the three-dimensional pressure gradient caused by the body shape and size. However, they revealed a flow topology slightly different from that proposed by previous researchers; the authors suggested that the difference observed might be due to a low accuracy of the experimental devices employed. Finally, in a recent paper of Siniša (2008), a study on the grid resolution and the inlet boundary conditions has been carried out using LES. Although a total of 15 million points were used, the resolution in the near-wall region which is important for LES is still not sufficiently fine. Two different inlet conditions have been used: a mean experimental profile and a timedependent boundary conditions produced with a precursor channel flow DNS at a lower Reynolds number. It was found that no significant improvement was obtained using the time-dependent inlet conditions.

In this work, a direct numerical simulation over an axisymmetric hill has been performed. To the author knowledge no previous simulation has been carried out with this approach because of the intensive computational cost involved. However, some simplifications have to be assumed: a lower Reynolds number of Re_H =6500 (i.e. 5% of the experimental conditions); Mach number $M_{\infty}=0.6$ in order to reduce the computational cost; periodic boundary conditions in the spanwise direction and inflow conditions generated by a precursor turbulent boundary layer simulation on a smaller domain.

6.2 Computational Domain and Flow Conditions

In Figure 6.1 is presented a sketch of the physical domain with the origin of the axis chosen at the center of the hill. The hill's shape is exactly the same of the experimental model used by Simpson *et al.* (2002), which is defined as:

$$\frac{y}{H} = -\beta \left[J_0(\Lambda) I_0\left(\frac{r\Lambda}{a}\right) - I_0(\Lambda) J_0\left(\frac{r\Lambda}{a}\right) \right],\tag{6.1}$$

where $\beta = 1/6.048$, $\Lambda = 3.1926$ and a = 2H is the radius of the circular base of the hill. J_0 is the Bessel function of first kind and I_0 the modified Bessel functions of first kind, respectively. The incoming boundary layer thickness δ is about half height of the hill and at $Re_{\delta^*} = 500$ the ratio between the boundary layer thickness δ and the boundary layer displacement thickness δ^* is about $\delta/\delta^* = 6.34$. Choosing δ^* as reference length, we have $H = 13\delta^*$.



Figure 6.1: Sketch of the hill physical domain.

Characteristic non-reflecting boundary conditions are applied at the top and the outlet of the domain, while the periodic boundary conditions are applied in the spanwise direction. Two different inlet boundary conditions have been tested: a) results from a precursor turbulent boundary layer fed into the domain (hereafter indicated as inflow of kind A); b) a turbulent boundary layer flow generated by a synthetic approach embedded in the simulation (hereafter indicated as inflow kind B). Four main simulations have been then carried out: (I) a coarse grid resolution with inflow condition of kind A; (II) a coarse grid resolution with inflow condition of kind B. The intentions of tests I and II were to verify the adequacy of the domain chosen and to validate the inflow condition of kind A.

The dimensions of the physical flow domains used are presented in Table 6.1 in comparison with some previous works. Streamwise and normal lengths are similar to that in the work of Wang *et al.* (2004) and Tessicini *et al.* (2007), while the length is slightly shorter in the spanwise dimension for tests I,II and III in order to reduce the computational cost. Uniform streamwise and spanwise grid spacing are used, while a hyperbolic sinusoidal stretching is applied in the wall-normal direction. The coarse grid simulations are under-resolved by a factor of 4 in both streamwise and spanwise directions and the first grid point is just above the sublayer region. In the tests III and IV a proper DNS resolution is achieved, at least on the top of the hill (see Table 6.2) where nine points are located in the sublayer $(y^+ = 10)$. The estimations of the resolution are based on the prescribed inflow profile.

Test Case	L_x	L_y	L_z	$d_i{}^a$
Ι	16H	3.205H	8.6H	4.4H
II	20H	3.205H	8.6H	4.4H
III	16H	3.205H	8.6H	4.4H
IV	20H	3.205H	10H	8.4H
Patel et al. (2003)	9.5H	3.205H	6H	3.4H
Wang et al. (2004)	16H	3.205H	11.67H	4.4H
Persson et al. (2007)	12H	3.205H	10H	3.4H
Tessicini et al. (2007)	16H	3.205H	11.67H	4.4H

^(a) d_i is distance from the inlet to the hill axis center.

Table 6.1: Dimensions of present and previous numerical works. H is the height of the hill.

Test Case	N_x	N_y	N_z	Δx^+	Δy^+_{min}	Δy_{max}^+	Δz^+
Ι	121	71	121	48.69	11.26	28.7	26.22
II	161	71	121	45.64	1.06	25.37	26.22
· III	481	161	481	12.17	1.06	25.37	6.55
IV	601	161	561	12.17	1.06	25.37	6.52

Table 6.2: Grid resolution of test cases presented.

6.3 Inflow Boundary Condition

6.3.1 Inflow Generation Methodology

As previously cited, two different kinds of inflow generation methods have been used, namely kind A, consisting in a precursor boundary layer developed on a minimal domain, and kind B, based on the digital filter idea of Klein *et al.* (2003), successively improved by Xie and Castro (2008) and later adapted to compressible flow by Touber and Sandham (2008). In kind B, the turbulent boundary layer properties (both mean velocity profile and Reynolds stress) are prescribed, along with artificially generated random numbers as white noise. A filter procedure is then used to retain the appropriate flow field values to reproduce the desirable turbulence correlations for given conditions. The advantage of this approach comparing to the synthetic method used by Sandham *et al.* (2003) is the elimination of spurious low frequencies in the outer region of the boundary layer, which can interfere with the flow separation process presented at the leeside of the hill. Details on the digital filter parameters can be found in the Appendix D and in Touber and Sandham (2008).

Kind A method consists in a precursor flat plate turbulent boundary layer simulation on a domain smaller than the hill domain. The instantaneous flow field obtained near the exit plane is saved in a time sequence of data slice and then fed into the large hill domain by time and space interpolations (see Figure 6.2). Periodic conditions are applied in the spanwise direction, where the length is large enough to contain all the turbulent scales of motion at a given Reynolds number. Non-reflecting boundary conditions are applied on the top and at the outlet of the domain, while the inlet condition is generated using a digital filter technique similar to inlet condition kind B.



Figure 6.2: Sketch of the interface precursor boundary layer hill system. The instantaneous flow field obtained near the exit plane is saved in a time sequence of data slice and then fed into the large hill domain by time and space interpolations.

6.3.2 Results of a Precursor Turbulent Boundary Layer

The precursor boundary layer simulation has a domain of $50\delta^*$, $10\delta^*$ and $8\delta^*$ in the streamwise (x), the wall normal (y) and the spanwise (z) directions, where the δ^* is the inlet displacement thickness of the boundary layer. A grid of $119 \times 61 \times 71$ points is used, uniformly distributed in the x and z directions and stretched in the y direction with a hyperbolic sinusoidal function. Based on the inflow conditions, the estimated grid resolutions are $\Delta x^+ = 11.09$, $\Delta z^+ = 6.24$ and the first point in the wall normal direction is about $\Delta y_1^+ = 0.99$ with a total of 10 points in the viscous sublayer region.

The precursor simulation starts with a uniform flow field and after an initial transient stage of about 100 time units (i.e. two through-flows), statistical samples are collected for every 100 time units until statistic convergence is achieved. This normally takes about 400 time units (i.e. 8 through-flows). The mean velocity u^+ , the turbulence intensities $(u_{rms}, v_{rms}, w_{rms})$ and the Reynolds stress $(\widetilde{u'v'})$ variations are shown in Figures 6.3 and 6.4, respectively, where δ is the boundary layer thickness and the variables are normalized by the friction velocity u_{τ} . The results are compared with the benchmark DNS data of Spalart (1988) at the same Reynolds number ($Re_{\delta}^* =$

500). It can be seen that the overall agreements are good to very good, with slightly over-prediction in the wake region of the mean velocity profile and the streamwise turbulence intensity u_{rms} . The simulation results at $x/\delta^* \simeq 45$ are saved at every 5-iteration intervals.



Figure 6.3: Mean streamwise velocity u^+ at $x/\delta^*=45$ for the precursor boundary layer. The data are compared with the DNS work of Spalart (1988) and the theoretical values of the law of the wall and log law.

6.4 Turbulent Flow Over the Hill: Coarse Grid Results

For both test cases I and II, after initial transient runs of about 300 time units, statistics data are collected for further 900 time units. Figure 6.5 gives the comparison of the separation bubble boundary lines. It can be seen that results from two tests agree well and small wiggles observed near the reattachment region are likely associated with the coarse grid resolution. This confirms that the feed-in technique works well and the same method will be used for the fine grid test IV described later.



Figure 6.4: Turbulence intensities at $x/\delta^*=45$ for the precursor boundary layer. The data are normalized by the friction velocity u_{τ} and compared with the DNS work of Spalart (1988).

Figure 6.6 shows the contours of the pressure coefficient C_p (defined as $C_p = (p_{l,s} - p_{r,s})/(p_{t,s} - p_{r,s})$, where $p_{l,s}$, $p_{r,s}$ and $p_{t,s}$ are the local, reference and total static pressure, respectively) on the wall surface. It can be seen that the C_p increases at the windward of the hill and decreases at the lee-side of the hill. Downstream in the wake region, the pressure recovers. The predicted minimum C_p happens around the top of the hill and, after about x/H = 2, it increases again with higher value in central area downstream of the hill. These features are in qualitatively good agreement with those observed by Simpson et al. (2002) in the high-Re experiments (see Figure 6.7).

The lee-side flow separation bubble can be seen in Figure 6.8, where the streamlines illustrate the flow re-circulation at the mid-plane z = 0. In this plane, the flow separation starts about x/H = 0.5 with a small closed bubble, followed by a more dominant large bubble with its center at about x/H = 2, y/H = 0.5. The flow reattaches at about x/H = 2.75. In the high-Re experiment, a small recirculation zone enclosing a very thin and elongated strip attached to the lee-side of the hill was observed. In comparison, the predicted bubble size is much larger, and this is probably



Figure 6.5: Separation bubble region: comparison between test case I (straight line), where the inlet is generated using a precursor boundary layer, and II (dashed line), where the inlet is generated using a synthetic approach embedded in the simulation.

due to Reynolds number effects.

Figure 6.9 illustrates the three-dimensional flow separations with the counter rotating vortex pair (CRVP) being captured on the lee-side of the hill. The topology analysis of flow separation and reattachment is also shown in this Figure, where two saddle points (S_1 and S_2) are identified at x/H=0.52, and x/H=2.8, at the mid-plane of z/H=0, along with two nodal (foci) separation points (N_{s1} and N_{s2}) at x/H=1.31, $z/H=\pm 0.8$ (i.e. the center of vortex core at the hill surface); two further nodal (foci) separation points (N_{s3} and N_{s4}) at x/H=0.34, $z/H=\pm 0.61$, and two further saddle points (S_3 and S_4) at x/H=0.34, $z/H=\pm 0.72$. In the high-Re experiment, only one pair of nodal and saddle separation points were identified. Here, two pairs of nodal and saddle separation points are captured in the low-Re simulation. The topology satisfies the well-known rule (Hunt *et al.*, 2006) as

$$\sum N - \sum S = 0. \tag{6.2}$$

where N and S are the number of nodal and saddle points, respectively.



Figure 6.6: Contours of pressure coefficient on the wall surface for the hill case. The circle of radius 2H indicates the location of the hill.



Figure 6.7: Pressure coefficient contours for the hill case from the experiment of Simpson *et al.* (2002). The circle of radius 2H indicates the location of the hill.



Figure 6.8: Streamlines of streamwise and vertical mean velocities for the hill case in the central plane (z=0).



Figure 6.9: Flow topologies for the hill case (coarse grid) in the near wall at $y^+=11.26$. The circle of radius 2H indicates the location of the hill.

6.5 Turbulent Flow Over the Hill: Fine Grid Results

The simulation on a fine grid (test IV) are presented with the following key results as: (a) mean profiles, (b) flow topology, (c) comparison with test case III, (d) turbulence intensity, (d) instantaneous flow field and coherent structures and (e) assessment of laminarisation.

6.5.1 Statistics of Mean Flow

After initial transient stage of 300 time units, statistical data are collected for a time period of 800 time units, with the convergence being verified and confirmed.

Figure 6.10 shows the computed mean surface pressure coefficient C_p defined as above for the coarse grid case. Similar to results on the coarse grid, the pressure



Figure 6.10: Mean surface pressure coefficient C_p for the hill case (fine grid) in the x - z plane. The four circle of diameter 0.5, 1, 1.5 and 2 indicates different levels of the hill height.

increases before approaching the hill and decreases over the crest of the hill. However, the pressure seems recovering downstream much later and two low pressure points can be identified on the leeward side at x/H = 1.56 and $z/H = \pm 0.74$. Moreover, there is a slightly higher pressure area just after the top of the hill which, as explained later, promote the fluid flow back toward the hill center and against the streamwise incoming current. From Figure 6.11, we can see that the pressure coefficient has a maximum value around 0.5 just before the hill (x/H=-2), it decreases to a minimum value around -0.3 just before the top of the hill. A value similar to the inlet it is only recovered at x/D = 4.14. The small increment of pressure just after the crest is also here clearly shown. On the same figure are shown the experimental data of Simpson *et al.* (2002): the separation is smaller and a stronger depression is shown.





The separation bubble is shown in Figure 6.12 with a vector plot of the velocity components. The dashed line refers to the test case (III) while the solid line to the test case (IV): the two lines are very close and this is a further validation of the inflow

methodology of kind B.

A close up figure on the upstream hill foot (i.e. Figure 6.13 a) shows a small recirculation region not being observed on a coarse grid simulation. This recirculation modifies the friction velocity profile along the streamwise direction and then affects the values of the normalized mean and turbulence intensity data. A close up on the leeward side of the hill (see Figure 6.13 b) also shows small secondary re-circulation bubble embedded inside a large primary re-circulation bubble. This bubble, although very small in size, seems to have some considerable influence on the flow separation behavior. In fact, this bubble, though not presented in the coarse grid simulation probably due to the lower resolution (around 8 points are presented in the fine grid test cases, while only 2 points in the coarse grid test cases), seems to push up the fluid flow coming from the top of the hill, causing an earlier separation of the boundary layer than that in the coarse grid case. The little increment of pressure above observed might be linked to the presence of this small bubble.



Figure 6.12: Separation bubble region: comparison between test case III (straight line) and IV (dashed line).

The mean streamwise velocity profile u^+ along the wall normal direction y^+ is shown in Figure 6.14. The boundary layer thickness is about $\delta = 6.34\delta^*$ at the inlet plane (x/H=-8.4 for test IV and x/H=-4.4 for test III). As cited above, different



(a) Upstream hill foot

(b) Lee-ward side of the hill

Figure 6.13: Secondary recirculation regions detected for the hill case on fine grid.

from the coarse grid, the presence of the small separation bubble at the foot of the windward of the hill at x/H = -2 causes the significant reduction of the friction velocity at x/H = -3 by almost half of the inflow value. Hence the mean velocity in the wall unit has been doubled at the edge of the boundary layer. At x/H = -1, the friction velocity recovers and increases, leading to a lower u^+ value there. Figure 6.15 presents the friction velocity normalized by the reference velocity u_{∞} along the streamwise direction. Where a negative value of $\partial \bar{u}/\partial y$ is encountered, the friction velocity is set to zero. This happens just before the hill foot x/H=-2 where the separation bubble is found and on on the leeside of the hill where the main separation occurs. At x/H = -1, the friction velocity recovers and increases, leading to a lower u^+ value there.

From Figure 6.16 to Figure 6.21 are presented the mean velocities values along the spanwise direction normalized by the reference velocity u_{∞} at four different locations: i) x/H=-2, upstream of the hill foot; ii) x/H=0, the top of the hill; iii) x/H=2, downstream hill foot and iv) x/H=4.14, the "mean" reattachment point.



Figure 6.14: Mean streamwise velocity u^+ along the vertical direction at different streamwise locations. The value refer to the middle plane (z = 0) and are compared with the law of the wall and log law.



Figure 6.15: Friction velocity along the streamwise direction. The values are normalized by the free stream velocity and the negative quantities have been cut off.



Figure 6.16: Mean streamwise velocity along the vertical direction for the hill case at (a) x/H=-2 and (b) x/H=0 at several spanwise locations.



Figure 6.17: Mean streamwise velocity along the vertical direction for the hill case at (a) x/H=2 and (b) x/H=4.14 at several spanwise locations.



Figure 6.18: Mean wall-normal velocity along the vertical direction for the hill case at (a) x/H=-2 and (b) x/H=0 at several spanwise locations.



Figure 6.19: Mean wall-normal velocity along the vertical direction for the hill case at (a) x/H=2 and (b) x/H=4.14 at several spanwise locations.



Figure 6.20: Mean spanwise velocity along the vertical direction for the hill case at (a) x/H=-2 and (b) x/H=0 at several spanwise locations.



Figure 6.21: Mean spanwise velocity along the vertical direction for the hill case at (a) x/H=2 and (b) x/H=4.14 at several spanwise locations.
6.5.2 Flow Topology

Figure 6.22 gives the streamlines from the fine grid simulation. Comparing to that of coarse grid, the separation bubble increases in size. The flow separation starts earlier at the crest of the hill and reattaches later at x/H=4.14. The re-circulation center is at x/H = 2.35, y/H = 1.06. The center of the small recirculation bubble which



Figure 6.22: Streamline for the hill case of normal and vertical velocity at the middle plane (z = 0) from fine grid simulation.

appears on the leeside of the hill is at x/H=0.63 and y/H=0.81. The flow topology analysis has identified four saddle points (see Figure 6.23) S_1 , S_2 , S_3 and S_4 along the mid-plane (z=0) at x/H=-2.4, x/H=-0.072, x/H=0.93 and x/H=4.25, respectively, and four nodal separation points (two of attachment and two of separation) N_{a1} , N_{a2} at mid-plane and x/H=-1.6, x/H=0.17, and N_{s1} and N_{s2} at x/H=1.53 and $z/H=\pm0.7$, respectively. As in the coarse grid case, the rule presented in eqn. 6.2 is satisfied.

The three-dimensional flow separation is well visualized from the streamline patterns in the near wall region as seen in Figure 6.24: the flow circulate around the hill and then it converge in a counter-rotating vortex pair (CRVP) with successively

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merging in the mean downstream current. An arbitrary mean velocity streamline is presented in Figure 6.25 and colored with the spanwise velocity. Following the streamline gives an idea of the 3D recirculation path: starting from point 1, the flow circulates around the hill until point 2 and then moves back towards the top (point 3) where it recirculates again on the leeside toward the bottom of the hill (point 4) and then merge with the streamwise current like in point 5.



Figure 6.23: Flow topologies for the hill case fine grid in the near wall at $y^+=1.06$.



Figure 6.24: Skin friction lines on the hill surface from fine grid simulation.



Figure 6.25: Arbitrary mean velocity streamline colored by the spanwise velocity.

6.5.3 Turbulence Intensity

Figures 6.26-6.29 gives the RMS turbulent intensities and the Reynolds stress (normalized by the friction velocity u_{τ} and averaged in the spanwise direction) at four successively streamwise locations: i) inlet (x/H=-8.4); ii) inlet for test III (x/H=-4.4); iii) reattachment point (x/H=4.14) and outlet of the domain x/H=11.6. At the inlet the predictions are in good agreement with the DNS of Spalart (1988), but downstream are slightly different from test case III, consistent with that seen in Figure 6.14. The reason for this is probably due to the rapid decrease of the friction velocity, as a result of the presence of a small separation bubble at x/H = -2. At the reattachment point, the values are much larger than that of inflow. The peak value for the u_{rms} quantities is not close to wall, probably due to the strong turbulence activity behind the hill due to the counter rotating vortex pair merging into the freestream. Finally, the RMS values at the exit of the domain are similar in magnitude to the incoming boundary layer, but the thickness of the boundary layer is much larger (nearly twice, i.e. equal to the hill's height).



Figure 6.26: Turbulence intensity and Reynolds shear stress at x/H=-8.4 along the vertical direction. The values are compared with the DNS work of Spalart (1988), normalized by the friction velocity u_{τ} and averaged in the spanwise direction.



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Figure 6.27: Turbulence intensity and Reynolds shear stress at x/H=-4.4 along the vertical direction. The values are compared with the DNS work of Spalart (1988), normalized by the friction velocity u_{τ} and averaged in the spanwise direction.





Figure 6.28: Turbulence intensity and Reynolds shear stress at x/H=4.14 along the vertical direction. The values are compared with the DNS work of Spalart (1988), normalized by the friction velocity u_{τ} and averaged in the spanwise direction.



Figure 6.29: Turbulence intensity and Reynolds shear stress at x/H=11.6 along the vertical direction. The values are compared with the DNS work of Spalart (1988), normalized by the friction velocity u_{τ} and averaged in the spanwise direction.

6.5.4 Instantaneous Flow Field and Coherent Structures

An instantaneous of the streamwise velocity at a middle plane of the spanwise direction (z/H=0) is shown in Figure 6.30. It is evident that the flow separates just behind the hill. The streamline visualization (Figure 6.31 and Figure 6.32) clearly shows multiple separation and reattachment points on the lee side with abrupt terminations which suggest instantaneous stagnation points. The red circle has radius 2H and indicate the position of the hill.



Figure 6.30: Mean streamwise velocity in the x-y plane for the hill case fine grid (z = 0) at $T = 1100\delta^*/u_{\infty}$.



Figure 6.31: Streamline in the x-y plane (z = 0) for the hill case fine grid at $T = 1100\delta^*/u_{\infty}$.

Figure 6.33 gives the fine grid flow structures of the same previous instantaneous flow field illustrated with the iso-surfaces of the second invariants criteria (see section 5.4) at the value of $\lambda_2 = 0.035$. Similarly, Figure 6.34 shows the coarse grid flow





Figure 6.32: Streamline in the x-y plane (z = 0) for the hill case fine grid at $T = 1100\delta^*/u_{\infty}$.

structures at the same value of $\lambda_2 = 0.035$. The effect of the hill seems to consist in a stretching of the structures along the streamwise direction. Considering Figure ??, after the hill the fluid is "pulled" in the streamwise direction by the freestream and upstream from the recirculation flow. Moreover, for the fine grid, it can be seen that the flow structures occurs in a larger area in the lee-side of the hill (consistent with larger separation bubble), but over the top of the hill they seem not presented. This indicates the flow may be partly re-laminarized in that region. For the coarse grid, the flow structures still exist around the crest of the hill.



Figure 6.33: Flow structures for the hill grid visualized using the criteria of second eigenvalue λ_2 of the sum of the square of the symmetric tensor $\overline{S_{ij}}$ and anti-symmetric tensor $\overline{\Omega_{ij}} = (1/2)(\partial \tilde{u}_i/\partial x_j - \partial \tilde{u}_j/\partial x_i)$ components of the gradient of the velocity field. The structures are shown for the value of $\lambda_2=0.035$ from the fine grid simulation (test IV).



Figure 6.34: Flow structures for the hill case at $\lambda_2=0.035$ from the coarse grid simulation (test *II*).

6.5.5 Re-laminarisation Issue

Due to a low Reynolds number used in the simulation and the strong favorable pressure gradient at the windward of the hill, re-laminarisation may occur. In a previous study by Sandham *et al.* (2003) for a turbulent flow over a bump geometry, the flow re-laminarisation issue has been studied by using a criteria on related to a general acceleration parameter (Jones and Launder, 1972) as

$$K = \frac{\nu_e}{U_e^2} \frac{\partial U_e}{\partial x} \tag{6.3}$$

where the 'e' represents the boundary layer edge and the criterion for flow laminarisation occurs roughly at $K > 3 \times 10^{-6}$. For the present hill simulation, the Kvariation is evaluated at the top boundary of the domain (i.e. y = 3.205H). The results shown in Figure 6.35) confirm that the K value has indeed exceeded the given laminarisation criterion. It is assumed that the flow laminarisation contributes to the larger separation bubble seen in Figure 6.12. From Figures 6.36 to 6.38 are shown the turbulent intensities on the windward side of the hill obtained without averaging the results in the spanwise direction. The values are reduced moving towards the hill confirming the laminarisation process.





Figure 6.35: Laminarisation parameter K variations along the streamwise location and middle plane (z = 0) at y = 3.205H.



Figure 6.36: Turbulent intensities for the hill case fine grid at x/H = -1.5. The values are not normalized.



Figure 6.37: Turbulent intensities for the hill case fine grid at x/H = -1. The values are not normalized.



Figure 6.38: Turbulent intensities for the hill case fine grid at x/H = -0.5. The values are not normalized.

6.6 Summary

Direct numerical simulation of a turbulent flow over an axisymmetric hill shape is performed. The simulation considers a low Reynolds number 500, which is about 1/20 of the experimental condition. Despite this difference, simulations are still able to capture the dynamic flow behavior and the key flow topologies in the separation region, in good qualitative agreement with the high-Re test. However, the predicted flow separation bubble at present low-Re seems much larger in size compared to that observed in the high-Re experiment. The simulation also reveals a small separation at the windward of the hill near the foot and a secondary separation bubble embedded inside a primary large bubble at the lee-side of the hill. It is found that flow laminarisation occurs in the vicinity of the crest of the hill. Consequently, this will alter the flow development and contribute to the earlier flow separation and the larger bubble. These findings are useful for future experimental validation study at low Reynolds number.

Chapter 7

Extension to Multiblock Code

In this Chapter we describe the extension of the full 3D SBLI code to a multiblock version. After a brief introduction on the advantages to deal complex geometries with more blocks (Section 7.1), details on the general structure of the new algorithm are given in Section 7.2. Then, a demonstration study (Section 7.3) is followed where results on a direct numerical simulation of a jet in a turbulent cross flow are presented. Section 7.4 contains a summary on the main features of the new multiblock SBLI code.

7.1 Advantages of a Multiblock Version

Many practical engineering problems occurs in complex geometries like engine combustion chamber, turbine blade, wing-body junction, etc. However, the body-fitted coordinate technique requires the use of structured mesh and some relatively simple geometries, like a jet in a cross flow system or a flow over a 2D square, normally require more than one block (see Figures 7.1a and 7.1b). Moreover, the meshes of different blocks need to satisfy the continuity at the interface between them. This makes complicated the grid generation and it explains the reason of the widely used finite volume methodology (where different meshes structured and unstructured can be applied) in today's commercial CFD software. However, as previously discussed in Chapter 1 of the thesis, the drawback of the finite volume method is the difficulty of its implementation for high-order numerical schemes. Moreover, the matching condition at the interface seems a good requirement but not absolutely necessary. In fact, the meshes from different blocks can also be just overlapped each other at the interface (these are commonly known as the Chimera grids) and the values at the different points can be found by an interpolation operation. This, obviously, introduce some disturbances and oscillations, and then some filtering scheme must be applied or some appropriate high-order interpolation schemes can be implemented. In this case, the drawback will be in a more intensive computational effort. Finally, the grid where the overlapping is obtained without any interpolation points can be an ideal solution for many kinds of geometry domains. However, the difficulty is to find an analytical function which satisfies two or more geometry domains at the same time.

A multiblock solver, based on the finite difference methodology, seems to be an ideal approach to study turbulent flows over complex geometries, provided the use of high-order schemes. The grid matching across the interfaces can be handled by using a global mapping for all the blocks. For example, polynomial functions can be used for the entire domain ensuring the continuity of the derivatives across the interfaces. The use of a global mapping for all the blocks will also simplify the interface issues presented in next section.

7.2 Structure of the Algorithm

When two or more blocks are used, it is necessary to resolve the interface issues between them, i.e. the information swap between the blocks. For example, we consider the case of a jet in a cross flow (Figure 7.1a) with a better local representation shown in Figure 7.2. It shows that block 1 and block 2 have a common interface layer at



Figure 7.1: Examples of multiblock geometries.

y = 0 in parallel to the x - z plane. Similar to the single block version of the code, the interface between two blocks can be considered like the interface between two processors when the parallel computing is involved. However, it is necessary to properly define the starting and finishing locations of the interfaces in terms of physical and computational coordinates. This operation can be done manually when the number of interfaces is small (1-4), but the definition process could become difficult if more blocks are involved. Moreover, for the same physical geometry, each grid resolution will have different bounds, making the use of the code not quite user friendly. The above problems can be addressed using a pre-processor program, where the geometry of the system and all the interfaces and block communication issues can be defined automatically, if a global mapping for all the blocks is employed. For example, in the jet in a cross flow case, each halo point of block 1 (marked with an empty black circle) is going to match its x, y, z coordinates with the corresponding points coordinates of block 2. Similar operation can be done for the halo points of block 2 presented in block 1 (marked with a red filled circle).

Considering that each block can be divided in more processors, the general steps of the pre-processor can be described as follows:



Figure 7.2: Interface points for the jet in cross flow system. The black circles indicate the halo points of block 1 presented in block 2, while the red filled circles indicate the halo points of block 2 presented in block 1.

- 1. The physical x, y, z coordinates values of each point of the domain are used for all blocks.
- 2. The boundary conditions for each block are assigned. The possible conditions are: "integral", "characteristic", "periodic", "interface" and "wall". Each block has six surfaces and the above conditions can be applied to each of these surfaces.
- 3. For each processor in a block, the pre-processor finds the relative physical and computational information: e.g. the number of points in each direction, indexes in the processor matrix, etc.
- 4. For each processor, each halo point of each processor belonging to a surface defined as "interface" is matched with the internal points of another processors. This operation defines the physical and computational bounds of the interfaces. Moreover, each block will be assigned with a set of six boundary conditions, one for each side. If only one processor is defined in a block, these boundary conditions are coincident with the block boundary conditions. If more than one processor is used in a block, the pre-processor will find the most appropriate boundary conditions considering the presence of interfaces between blocks. For



Figure 7.3: Boundary conditions after splitting between two processors.

example, we consider a 2D block with the following boundary conditions (see Figure 7.3):

- left side: integral
- right side: characteristic
- bottom side: wall
- upper side: wall

Suppose now to divide the block with two processors along the x direction. The new boundary conditions for each processor will be:

for processor 1)

- left side: integral
- right side: contiguous
- bottom side: wall
- upper side: wall

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for processor 2)

- left side: contiguous
- right side: characteristic
- bottom side: wall
- upper side: wall

The new label "contiguous" indicates the absence of a physical boundary and the presence of an adjacent processor.

If an interface is present, the pre-processor will define a new type of interface over the range of overlapped halo points between the adjacent blocks. For example, in the case of the jet in a cross flow described above, if one processor is used for each block then the boundary conditions will be (see Figure 7.4):

processor 1)

- left side: integral
- right side: characteristic
- bottom side: interface-mix
- upper side: wall

processor 2)

- left side: integral
- right side: characteristic
- bottom side: wall
- upper side: interface-full



Figure 7.4: Boundary conditions for the jet in a cross flow system.

The "interface-mix" condition indicates the presence of some halo points (but not all) in another block (block 2 in this case), while the condition "interfacefull" indicates that all the halo points are in the other block (block 1 in this case). This difference is necessary in order to speed up the derivative calculation across the interface. Ideally, each processor should have an "interface-full" if an interface occurs.

5. Mesh data are saved in the mesh files, while all the information about number of processors, blocks, partitioning, interface bounds, etc. are saved in the file called 'locate.dat'.

For very large job the pre-processor program can require huge amount of memory and computational time. A parallel version of the preprocessor will be an ideal solution to these cases.

7.3 Demonstration: DNS of a Jet in a Cross Flow

In order to demonstrate the multiblock version of the code, a direct numerical simulation of a jet in a cross flow (here after JICF) has been carried out. Moreover, this kind of flow configuration has many practical applications like cooling film for turbine blade, fuel injection in combustion chamber, dispersion of polluting in the atmosphere, etc. High fidelity numerical simulation can be useful to understand the mixing mechanism which occurs downstream the jet orifice.

7.3.1 Review of Previous Works

The JICF have been investigated experimentally and numerically by many researchers, however here we mainly reference to two studies: Ajersch *et al.* (1997) and Sau *et al.* (2004). In the former, detailed measurements and numerical simulations of multiple jets in a cross flow are presented. The CFD code used is a RANS solver based on the $k - \epsilon$ model corrected with a special near-wall treatment. The Reynolds number, based on the jet diameter and freestream quantities, was 4700. The jet to cross flow velocity ratios examined were 0.5, 1.0 and 1.5. Good agreement has been found between experiments and simulations for the nonuniform mean flow at the jet exit plane, while the velocities and stresses on the jet centerline downstream of the orifice are less well predicted. In the second work, simulations were performed for two moderate values of the Reynolds number 225 and 300, based on the jet to cross flow velocity ratio, 2.5 and 3.4.

In both cases, from a physical point of view, a counter rotating vortex pair is observed for a sufficiently high jet to cross flow momentum ratio $(R = V_j/V_{cf})$ and a backflow region is present just downstream of the jet. Three momentum ratio were examined: 0.5, 1.0 and 1.5 at a jet Reynolds number approximately 4700.

Some interesting physical flow phenomena have been found in the work of Sau *et al.* (2004), where a direct numerical simulation of a square jet in a laminar cross flow has been carried out. Moreover, an upstream horse-shoe vortex system is detected, which is the results of the interaction between the incoming channel floor shear layer and the transverse jet. Two moderate Reynolds numbers, 225 and 300, (based on the jet width and the average cross-flow inlet velocity), have been simulated for two different jet to cross flow velocity ratios, 2.5 and 3.5.

7.3.2 Domain, Grid and Flow Conditions

A sketch of the system can be seen in Figure 7.1a. Two different Reynolds numbers have been tested ($Re_{\delta^*} = 1000$ and $Re_{\delta^*} = 2000$, based on the inlet boundary layer displacement thickness δ_{in}^* and the free stream propriety of the cross flow) for two different jet inflow profiles (a fully developed and a constant velocity profile) and three different jet to cross flow velocity ratios (0.5, 1.0 and 1.5). Tables 7.1 and 7.2 give the dimensions, the number of points and grid resolutions for all the tests cases. We consider a subsonic flow at $M_{\infty} = 0.6$.

Re_{δ^*}	L_x	L_y	L_z	N_x	Ny	N _z	Δx^+	Δy^+	Δz^+
1000	22.73	9.47	16	221	81	151	1.03 - 21.06	0.99 - 15.86	0.99 - 10.27
2000	18.35	4.96	8	401	91	251	1.12 - 20.48	1.12 - 28.16	1.12 - 6.48

Table 7.1: Cross flow domain parameters

The periodic conditions are applied in the spanwise direction, so the simulation is similar to a series of parallel jets. However, because we are interested in only one single jet flow dynamics, we have chosen a domain wider enough to avoid interactions in the spanwise direction. The inflow cross flow profile is provided by a precursor

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Re_{δ^*}	L_x	L_y	L_z	N _x	N_y	N_z	Δx^+	Δy^+	Δz^+
1000	1	1	1	41	31	31	1.03 - 21.06	0.99 - 15.86	0.99 - 10.27
2000	1	$1-5^{a}$	1	81	41	51	1.12 - 20.48	1.12 - 28.16	1.12 - 6.48

^a for constant jet inlet profile

Table 7.2: Jet domain parameters

turbulent boundary layer similar to that in the hill case (see section 6.2) and the inlet pressure field is obtained by an extrapolation condition. Characteristic non-reflecting boundary conditions are applied at the outflow and at the top of the domain (e.g. block 1). For the jet inflow (i.e. block 2) two different profiles have been used: a) similar to the work of Sau *et al.* (2004), a fully developed axial inlet velocity (White, 1991) described as:

$$v(x,z) = \frac{48}{\alpha \pi^3} \beta(x,z) \tag{7.1}$$

where

$$\alpha = 1 - \frac{192}{\pi^5} \sum_{i=1,3,5,\dots}^{\infty} \frac{tanh(\xi)}{i^5}$$
(7.2)

$$\beta(x,z) = \sum_{i=1,3,5...}^{\infty} (-1)^{(i-1)/2} \left(1 - \frac{\cosh\left[(2h-1)\xi\right]}{\cosh(\xi)} \right) \frac{\cos(2x\xi)}{i^3}$$
(7.3)

$$\xi(i) = \frac{\pi i}{2d} \tag{7.4}$$

b) a constant profile.

Due to the presence of the jet, a non uniform mesh has been adopted in all directions: in the streamwise direction a cubic function is used on the left and the right sides of the jet while a fifth-order polynomial function has been used for the jet center region (see Figure 7.5 a). This guarantees the continuity of the first and the second derivatives across the whole domain. Similar stretching function has been applied in the spanwise direction, but the extremes of the domain are stretched with a fifth-order polynomial function in order to guarantee the continuity of the derivatives considering that periodic conditions are applied in this direction. A cubic function stretching is applied in the normal direction (see Figure 7.5 b).





(a) Fifth order polynomial function

(b) Cubic function

Figure 7.5: Fifth and cubic function for mesh stretching in the jet in cross flow system along the streamwise and normal directions.

7.3.3 Results

In the following are presented some of the results from the different tests cases cited above. A comparison with the experimental results of Ajersch *et al.* (1997) is presented, but no details on the flow dynamic and topology are here given.

The data have been collected after a transient time of 100 time units: initially run 50 time units without the jet block, in order to fully develop the turbulent flow from the inlet, and then run another 50 time units with the jet block turned on. After the transient time, the data are collected for further successive 200 time units until statistical convergence is achieved.

Figures 7.6a and 7.6b show the instantaneous flow field of the normal velocity after 200 time units for the $Re_{\delta^*} = 1000$ test case with fully developed mean jet profile at the inlet and a jet to cross flow velocity ratio equals to 1.5. Streamlines are also represented. The jet penetration is around 2 jet exit diameters in the normal direction. A backflow region is detected downstream of the jet exit, while the horseshoe vortex system, or recirculation zone, is captured upstream of the jet exit. Figure 7.7 and Figure 7.8 are shown the statistics data in the x - z plane at different ylocations. It is noted that the formation of a counter rotating vortex pair behind the jet exit which increases while moving away from the wall to a maximum around $y^+ = 16.23$ and then disappears in the streamwise current. The flow topology is similar to those presented in the work of Sau *et al.* (2004) and Ajersch *et al.* (1997).

Finally, Figures 7.9-7.13 show the results at $Re_{\delta^*} = 1000$ for the mean velocities \tilde{u}, \tilde{v} and \tilde{w} and for the mean shear stress $\tilde{u'v'}$ and turbulence kinetic energy k at different streamwise locations, compared with the results from the work of Ajersch *et al.* (1997). Data are plotted at a cross x - y plane at z = 0 and normalized by the free-stream velocity \tilde{u}_{∞} . General good agreements are showed for the streamwise and the spanwise velocities, while the normal velocity profile has some incongruence especially at the jet exit. This is probably due to the different jet profile used. The

discrepancies are also found in the turbulence kinetic energy and also in the shear stress profiles.



(b) Close up

Figure 7.6: Instantaneous normal velocity and streamlines in the middle plane (z = 0) for the jet in cross flow after 200 time units.



Figure 7.7: Mean normal velocity and streamlines in the middle plane (z = 0) for the jet in cross flow.



Figure 7.8: Mean normal velocity and streamlines in the middle plane (z = 0) for the jet in cross flow.



Figure 7.9: Comparison of the mean streamwise velocity \tilde{u}/u_{∞} at $y/\delta = 0$ with the work of Ajersch *et al.* (1997) at several streamwise locations.



Figure 7.10: Comparison of the mean normal velocity \tilde{v}/u_{∞} at $y/\delta = 0$ with the work of Ajersch *et al.* (1997) at several streamwise locations.



Figure 7.11: Comparison of the mean spanwise velocity \tilde{w}/u_{∞} at $y/\delta = 0$ with the work of Ajersch *et al.* (1997) at several streamwise locations.



Figure 7.12: Comparison of the mean shear stress \tilde{uv}/u_{∞}^2 at $y/\delta = 0$ with the work of Ajersch *et al.* (1997) at several streamwise locations.



Figure 7.13: Comparison of turbulence kinetic energy velocity k/u_{∞}^2 at $y/\delta = 0$ with the work of Ajersch *et al.* (1997) at several streamwise locations.

7.4 Summary

A multiblock version of the code has been developed and described. The main modification is the implementation of a pre-processor program which generates the mesh and automatically finds all the information for each block. The new version of the code has been successfully tested for a jet in a cross flow system at different Reynolds numbers and different jet to cross flow velocity ratios (R), with representation results here presented for $Re_{\delta^*} = 1000$ and R = 1. However, further validation could be carried out for different systems and flow conditions.

Chapter 8

Conclusions and Future Works

The SBLI code has been successfully extended to have fully 3D curvilinear capability. Validation tests and direct numerical simulations of turbulent flows in a ridged channel and over an axisymmetric hill have been performed. Moreover, a multiblock version of the code is developed and results from a direct numerical simulation of a jet in a cross flow are presented. In this Chapter we will summarize the main findings of the thesis and provide suggestions on possible future works.

8.1 Conclusions

8.1.1 Extension to Fully 3D Curvilinear Geometries

Following a different formulation than Visbal and Gaitonde (2001) for the inviscid terms of the Navier-Stokes equation, a fully 3D version of the SBLI code has been successfully developed and tested for different cases with the following main findings:

- preservation on a wavy grid has been proven with an error of 1×10^{-16} (same order of the machine-precision);
- the errors found in a laminar channel flow on a highly distorted grid are pre-
served when the flow direction and the position of the walls are altered along each direction;

• a pre-compiler program has been developed in order to simplify the r.h.s. subroutine in the presence of non curvilinear geometries.

8.1.2 Turbulent Flow in a Ridged Channel

DNS of a turbulent flow in a ridged channel has been carried out at two Reynolds numbers: $Re_{\tau} = 150$ and $Re_{\tau} = 360$. The data have been compared with previous numerical and experimental results and the following are the main findings:

- at a low Reynolds number ($Re_{\tau} = 150$), the mean quantities agrees fairly well with previous studies and two large recirculations are observed in the transverse plane. The primary wall stress agrees as well;
- the turbulence intensity at the extreme locations of the domain is in better agreement with that from turbulent plane channel flow than that presented by Kawamura and Sumori (1999) and FA;
- at a medium Reynolds number ($Re_{\tau} = 360$) the effect of the ridge on the streamwise flow is stronger at the ridge center and weaker at the extreme locations. Consistently with this, at the extreme locations the turbulence intensity is approaching to the plane channel flow. Moreover, a small secondary recirculation has been captured at the ridge foot areas;
- differently from the FA work where additional small recirculations seems not to affect the main flow, here a variation in the streamwise velocity profile is observed. This seems consistent with other findings, considering that the streamwise velocity has low intensity in this region where the recirculation occurs and

then the value of the secondary recirculation, even if small as well, is "large" enough to affect the streamwise velocity profile.

8.1.3 Turbulent Flow over an Axisymmetric Hill

DNS of a turbulent flow over an axisymmetric hill at $Re_{\delta^*} = 500$ has been carried out. This is the main simulation of this thesis work with the following findings:

- on a coarse grid, simulation gives results in good qualitatively agreement with experimental data obtained at a high Reynolds number. The flow accelerates over and around the hill of it and then it separates on the leeside. A counter rotating vortex pair merging in the free stream flow has been detected. The flow topology analysis shows three saddle separation points on the middle plane (z/H = 0) and three nodal separation points on the leeside. A small (clockwise) recirculation bubble embedded inside the main recirculation has been detected just after the top of the hill. However, considering the findings from a fine grid simulation, we only retain the coarse grid results "apparently correct";
- on a fine grid, the main separation bubble detected is larger than that from a coarse grid simulation. The effect is due to the presence of a small (anticlockwise) recirculation bubble closer to the top of the hill which pushes up the incoming streamwise flow and causes an earlier separation of the boundary layer. The reason is probably due to an improved grid resolution which allows to correctly resolve the complex separation which occurs on the leeside. The flow topology analysis shows the presence of four separation saddle points and four nodal points (two of attachment and two foci of separation);
- the coherent structures analysis shows the disappearance of finer flow structures over the hill. A laminarisation effect is suggested and also confirmed by the high laminarisation parameter evaluated in this region;

- a small recirculation bubble has been detected at the foot upstream of the hill. Its presence modifies the local friction velocity value with a drastic reduction and thus leads to a high value of the turbulence intensities;
- a new inflow generation method, based on a precursor boundary layer simulation, has been successfully tested. The main advantage is the reduction of the computational domain and thus the costs. The new method, based on a spanwise duplication of the precursor boundary layer simulation, gives similar separation behavior behind the hill on the coarse resolution and reasonable good results on the fine grid.

8.1.4 Extension to Multiblock Version

A new multiblock version of the SBLI code has been presented. The main feature is the implementation of a pre-processor program which defines all the computational parameters for a given system (e.g. the number of points, blocks, indexes, etc.) and automatically finds the computational bounds of each interface between adjacent blocks. The multiblock version has been tested for a jet in a cross flow system at different flow conditions. Satisfactory results have been achieved in comparison with available experimental data.

8.2 Future Work

The new fully 3D curvilinear and multiblock version of the SBLI code is still in its early development stage. It is possible that ion depth bugs and incompleteness still remains and more validation tests and applications are desiderable to make the code more robust. Thus, the following tasks are suggested:

• LES module needs to be tested and possible expanded to include other subgrid

models;

- TVD scheme for shock capturing needs to be tested;
- further optimization of the version;
- MPI-IO implementation and scalability test at large scale and on different High Performance Computing (HPC) platforms.

Concerning the three-dimensional boundary layer separation of the hill problem, the laminarisation at a low Reynolds number has made a drastic change of the flow topology and makes a comparison with a high Reynolds number very difficult. Then, the following points are suggested for future investigation:

- a DNS at a higher Reynolds number (for example $Re_{\delta^*} = 1000$) should be high enough to avoid the flow laminarisation which unfortunately occurs at a lower $Re_{\delta^*} = 500$;
- LES at same Reynolds number used in the experimental study should be able to capture the main of the features presented, provided that a high resolved LES can be carried out at reasonable affordable cost;
- the effect of different hill height or ratio δ/H can considerably improve in the understanding of the three-dimensional flow separation mechanism;
- new RANS models can be implemented, tested and compared with the present DNS.

Appendix A

Dimensionless N-S Equations and Other Useful Equations

A.1 Derivation of Dimensionless N-S Equations

We present here the mathematical procedures to obtain the dimensionless NS equations based on the dimensional eqns. (2.1)-(2.3) and the reference parameters presented in eqn. (2.10).

By substituting the correspondent dimensional variables, we have

Continuity equation

$$\frac{\rho_r^* U_r^*}{\delta_r^*} \frac{\partial \rho}{\partial t} + \frac{\rho_r^* U_r^*}{\delta_r^*} \frac{(\partial \rho u_i)}{\partial x_i} = 0,$$

or

$$\frac{\partial \rho}{\partial t} + \frac{(\partial \rho u_i)}{\partial x_i} = 0. \tag{A.1}$$

Appendix A. Dimensionless N-S Equations and Other Useful Equations

Momentum equation

The first step is to derive the dimensionless form of the stress term as

$$\tau_{ij}^* = \mu \mu_r^* \frac{U_r^*}{\delta_r^*} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) = \mu_r^* \frac{U_r^*}{\delta_r^*} \tau_{ij}',$$

where

$$\tau_{ij}' = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right).$$

By substituting these in the momentum equations we obtain

$$\frac{\rho_r^* U_r^{*2}}{\delta_r^*} \frac{\partial(\rho u_i)}{\partial t} + \frac{\rho_r^* U_r^{*2}}{\delta_r^*} \frac{\partial(\rho u_i u_j + p\delta_{ij})}{\partial x_j} - \mu_r^* \frac{U_r^*}{\delta_r^{*2}} \frac{\partial \tau_{ij}'}{\partial x_j} = 0,$$

or

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j + p\delta_{ij})}{\partial x_j} - \frac{\delta_r^*}{\rho_r^* U_r^{*2}} \mu_r^* \frac{U_r^*}{\delta_r^{*2}} \frac{\partial \tau_{ij}'}{\partial x_j} = 0.$$

Because

$$\frac{\delta_r^*}{\rho_r^* U_r^{*2}} \mu_r^* \frac{U_r^*}{\delta_r^{*2}} = \frac{\mu_r^*}{\rho_r^* U_r^* \delta_r^*} = \frac{1}{Re},$$

where the Reynolds number is defined as

$$Re = \frac{\rho_r^* U_r^* \delta_r^*}{\mu_r^*}.$$

We now define

$$\tau_{ij} = \frac{\tau'_{ij}}{Re} = \frac{\mu}{Re} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right),$$

and find dimensional form of the

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j + p\delta_{ij})}{\partial x_j} - \frac{\partial\tau_{ij}}{\partial x_j} = 0.$$
(A.2)

Energy equation

As before, we perform the dimensionless of the following terms

$$E_{tot}^{*} = \rho^{*} \left(e^{*} + \frac{1}{2} u_{i}^{*} u_{i}^{*} \right) = \rho_{r}^{*} U_{r}^{*2} \rho \left(e + \frac{1}{2} u_{i} u_{i} \right) = \rho_{r}^{*} U_{r}^{*2} E_{tot},$$
$$q_{i}^{*} = -k \frac{T_{r}^{*}}{\delta_{r}^{*}} \frac{\partial T}{\partial x_{i}}.$$

By substituting them into the energy equation, we have

$$\frac{\rho_r^* U_r^{*3}}{\delta_r^*} \frac{\partial E_{tot}}{\partial t} + \frac{\rho_r^* U_r^{*3}}{\delta_r^*} \frac{\partial [(E_{tot} + p)u_i]}{\partial x_i} - k \frac{T_r^*}{\delta_r^{*2}} \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i}\right) - \frac{\mu_r^* U_r^{*2}}{\delta_r^{*2}} \frac{\partial u_j \tau_{ij}'}{\partial x_i} = 0,$$
$$\frac{\partial E_{tot}}{\partial t} + \frac{\partial [(E_{tot} + p)u_i]}{\partial x_i} - \frac{\delta_r^*}{\rho_r^* U_r^{*3}} k \frac{T_r^*}{\delta_r^{*2}} \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i}\right) - \frac{\delta_r^*}{\rho_r^* U_r^{*3}} \frac{\mu_r^* U_r^{*2}}{\delta_r^{*2}} \frac{\partial u_j \tau_{ij}'}{\partial x_i} = 0.$$

The coefficient in the last terms can be simplified as

$$\frac{\delta_r^*}{\rho_r^* U_r^{*3}} \mu_r^* \frac{{U_r^*}^2}{{\delta_r^*}^2} = \frac{\mu_r^*}{\rho_r^* U_r^* \delta_r^*} = \frac{1}{Re}$$

and

$$\frac{\delta_r^*}{\rho_r^* U_r^{*3}} k \frac{T_r^*}{\delta_r^{*2}} = \frac{\mu_r^*}{\mu_r^*} \frac{1}{\rho_r^* U_r^* \delta_r^*} \frac{k}{U_r^{*2}} T_r^* = \frac{1}{Re} \frac{k}{\mu^*} \frac{\mu}{U_r^{*2}} T_r^*.$$

If we rewrite the U_r^{*2} in this way

$$U_r^{*2} = M^2 a_r^{*2} = M^2 \gamma \underbrace{C_v(\gamma - 1)}_R T_r^* = M^2 C_p(\gamma - 1) T_r^*,$$

and we substituting in the precedent equation, we have

$$\frac{1}{Re}\frac{k}{\mu^*}\frac{\mu}{U_r^{*2}}T_r^* = \frac{1}{Re}\frac{k}{\mu^*}\frac{\mu T_r^*}{M^2 C_p(\gamma-1)T_r^*} =$$

$$\frac{1}{Re} \frac{k}{\mu^* C_p} \frac{\mu T_r^*}{M^2 (\gamma - 1) T_r^*} = \frac{1}{Re} \frac{1}{Pr} \frac{\mu}{M^2 (\gamma - 1)},$$

where

$$\frac{1}{Pr} = \frac{k}{\mu^* C_p}.$$

Now, posing

$$q_i = -\frac{1}{Re} \frac{1}{Pr} \frac{\mu}{M^2(\gamma - 1)} \frac{\partial T}{\partial x_i},$$

the final dimensional form of the energy equation is

$$\frac{\partial E_{tot}}{\partial t} + \frac{\partial [(E_{tot} + p)u_i]}{\partial x_i} + \frac{\partial q_i}{\partial x_i} - \frac{\partial u_j \tau_{ij}}{\partial x_i} = 0.$$
(A.3)

Appendix A. Dimensionless N-S Equations and Other Useful Equations

A.2 Others Useful Formulas

In following are presented some others useful formulas obtained for perfect gas and used in the code:

$$e = \frac{T}{(\gamma - 1)\gamma M^2},\tag{A.4}$$

that is correct because:

$$M^{2} = \frac{U_{r}^{*2}}{a_{r}^{*2}} = \frac{U_{r}^{*2}}{\gamma R T_{r}^{*}},$$

then

$$\frac{T}{(\gamma-1)\gamma M^2} = \frac{T}{(\gamma-1)\gamma \frac{U_r^{*2}}{\gamma R T_r^*}} = \frac{T T_r^*}{(\gamma-1)\frac{U_r^{*2}}{R}} = \frac{T^* R}{\frac{R}{C_v} U_r^{*2}} = \frac{T^* C_v}{U_r^{*2}} = \frac{e^*}{U_r^{*2}} = e.$$

Moreover:

$$p = \frac{p^*}{p_r^*} = \frac{\rho^* RT^*}{\rho_r^* U_r^{*2}} = \frac{\rho \rho_r^* RTT_r^*}{\rho_r^* U_r^{*2}} = \frac{\rho T RT_r^*}{U_r^{*2}} = \frac{\rho T}{\gamma M^2}.$$
 (A.5)

Appendix B

N-S Equations in 3D Fully Curvilinear Coordinates

In this Appendix we give the mathematical manipulations to obtain the eqn. (2.33). We start from the general form of transformation of eqn. (2.18):

 $\frac{\partial U}{\partial t} + \frac{\partial F}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial F}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial G}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial G}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial G}{\partial \zeta} \frac{\partial \zeta}{\partial y} + \frac{\partial H}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial H}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial H}{\partial \zeta} \frac{\partial \zeta}{\partial z} = 0.$ (B.1)

If we multiply each terms by Jcobian quantity J defined in section 2.2, we have

$$J\frac{\partial U}{\partial t} + J\frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial x} + J\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial x} + J\frac{\partial F}{\partial \zeta}\frac{\partial \zeta}{\partial x} + J\frac{\partial G}{\partial \xi}\frac{\partial \xi}{\partial y} + J\frac{\partial G}{\partial \eta}\frac{\partial \eta}{\partial y} + J\frac{\partial G}{\partial \zeta}\frac{\partial \zeta}{\partial y} + J\frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial z} + J\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial z} + J\frac{\partial H}{\partial \zeta}\frac{\partial \zeta}{\partial z} = 0.$$
(B.2)

We can rewrite the first derivative term in the x-direction of F as:

$$J\frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial \xi}\left(JF\frac{\partial \xi}{\partial x}\right) - F\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right).$$

Similarly we can rewrite all other terms in this way and we substitute them in eqn. (B.1) we have

$$J\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi} \left(JF\frac{\partial \xi}{\partial x}\right) - F\frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta} \left(JF\frac{\partial \eta}{\partial x}\right) - F\frac{\partial}{\partial \eta} \left(J\frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial \zeta} \left(JF\frac{\partial \zeta}{\partial x}\right) - F\frac{\partial}{\partial \zeta} \left(J\frac{\partial \zeta}{\partial x}\right) + \frac{\partial}{\partial \zeta} \left(JG\frac{\partial \zeta}{\partial y}\right) - G\frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta} \left(JG\frac{\partial \eta}{\partial y}\right) - G\frac{\partial}{\partial \eta} \left(J\frac{\partial \eta}{\partial y}\right) + \frac{\partial}{\partial \zeta} \left(JG\frac{\partial \zeta}{\partial y}\right) - G\frac{\partial}{\partial \zeta} \left(J\frac{\partial \zeta}{\partial y}\right) + \frac{\partial}{\partial \zeta} \left(JG\frac{\partial \zeta}{\partial y}\right) - G\frac{\partial}{\partial \zeta} \left(J\frac{\partial \zeta}{\partial y}\right) + \frac{\partial}{\partial \zeta} \left(JG\frac{\partial \zeta}{\partial y}\right) - G\frac{\partial}{\partial \zeta} \left(J\frac{\partial \zeta}{\partial z}\right) - G\frac{\partial}{\partial \zeta} \left(J\frac{\partial \zeta$$

Now we want to simplify this equation. If we combine all the terms with common F, we have:

$$-F\left[\frac{\partial}{\partial\xi}\left(J\frac{\partial\xi}{\partial x}\right) + \frac{\partial}{\partial\eta}\left(J\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial\zeta}\left(J\frac{\partial\zeta}{\partial x}\right)\right]$$

We can prove that the quantity in the bracket is equivalent to zero by using the metrics eqn. (2.32):

$$\frac{\partial}{\partial\xi} \left(\frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \eta} \right) + \frac{\partial}{\partial\eta} \left(-\frac{\partial y}{\partial\xi} \frac{\partial z}{\partial \zeta} + \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \xi} \right) + \frac{\partial}{\partial\zeta} \left(\frac{\partial y}{\partial\xi} \frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \xi} \right) = \\ \frac{\partial^2 y}{\partial \eta \partial \xi} \frac{\partial z}{\partial \zeta} + \frac{\partial y}{\partial \eta} \frac{\partial^2 z}{\partial \zeta \partial \xi} - \frac{\partial^2 y}{\partial \zeta \partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \zeta} \frac{\partial^2 z}{\partial \eta \partial \xi} \\ - \frac{\partial^2 y}{\partial \xi \partial \eta} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \xi} \frac{\partial^2 z}{\partial \zeta \partial \eta} + \frac{\partial^2 y}{\partial \zeta \partial \eta} \frac{\partial z}{\partial \xi} + \frac{\partial y}{\partial \zeta} \frac{\partial^2 z}{\partial \xi \partial \eta} \\ + \frac{\partial^2 y}{\partial \xi \partial \zeta} \frac{\partial z}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial^2 z}{\partial \eta \partial \zeta} - \frac{\partial^2 y}{\partial \eta \partial \zeta} \frac{\partial z}{\partial \xi} - \frac{\partial y}{\partial \eta} \frac{\partial^2 z}{\partial \xi}.$$

The sum of the term with same underline has equal zero if the function are continuous second-order derivable (*Schwartz's theorem* on the cross derivatives). Same can be done for the G and H terms. Then, we can rewrite the final eqn. (B.1) as

$$J\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi} \left(JF\frac{\partial \xi}{\partial x} + JG\frac{\partial \xi}{\partial y} + JH\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \eta} \left(JF\frac{\partial \eta}{\partial x} + JG\frac{\partial \eta}{\partial y} + JH\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(JF\frac{\partial \zeta}{\partial x} + JG\frac{\partial \zeta}{\partial y} + JH\frac{\partial \zeta}{\partial z} \right) = 0,$$

or, similar to the eqn. (2.33) as

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial \xi} + \frac{\partial G_1}{\partial \eta} + \frac{\partial H_1}{\partial \zeta} = 0, \tag{B.3}$$

where

$$U_1 = JU, \tag{B.4}$$

$$F_1 = \left(JF\frac{\partial\xi}{\partial x} + JG\frac{\partial\xi}{\partial y} + JH\frac{\partial\xi}{\partial z}\right), \tag{B.5}$$

$$G_1 = \left(JF\frac{\partial\eta}{\partial x} + JG\frac{\partial\eta}{\partial y} + JH\frac{\partial\eta}{\partial z}\right), \qquad (B.6)$$

$$H_1 = \left(JF\frac{\partial\zeta}{\partial x} + JG\frac{\partial\zeta}{\partial y} + JH\frac{\partial\zeta}{\partial z}\right). \tag{B.7}$$

Appendix C

Nomenclature used in the Full 3D Version of the SBLI Code

In the following we present the equations used in the SBLI code divided in: inviscid terms, viscous momentum terms, heat conductive terms and viscous energetic terms. For the viscous momentum terms we have used this nomenclature: Appendix C. Nomenclature used in the Full 3D Version of the SBLI Code

C.1 Nomenclature used for the Jacobian Terms

The inverse of the matrix presents in the eqn. (2.29) is indicates as

$$\begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} & \frac{\partial\xi}{\partial z}, \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} & \frac{\partial\eta}{\partial z}, \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z}, \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\zeta}, \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} & \frac{\partial y}{\partial\zeta}, \\ \frac{\partial z}{\partial\xi} & \frac{\partial \zeta}{\partial\eta} & \frac{\partial \zeta}{\partial\zeta}, \end{pmatrix}^{-1} = \frac{\begin{pmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\zeta}, \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\eta} & \frac{\partial z}{\partial\zeta}, \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\eta} & \frac{\partial z}{\partial\zeta}, \end{pmatrix}^{-1} = \frac{\begin{pmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\zeta}, \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} & \frac{\partial y}{\partial\zeta}, \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\eta} & \frac{\partial z}{\partial\zeta}, \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\eta} & \frac{\partial z}{\partial\zeta} \end{pmatrix}^{-1}$$

where:

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix}.$$
 (C.1)

and

$$Jac_{11} = \left(\frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta}\frac{\partial z}{\partial \eta}\right), \quad Jac_{12} = -\left(\frac{\partial x}{\partial \eta}\frac{\partial z}{\partial \zeta} - \frac{\partial x}{\partial \zeta}\frac{\partial z}{\partial \eta}\right), \quad Jac_{13} = \left(\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \eta}\right),$$

Appendix C. Nomenclature used in the Full 3D Version of the SBLI Code

$$Jac_{21} = -\left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta}\frac{\partial z}{\partial \xi}\right), \quad Jac_{22} = \left(\frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \zeta} - \frac{\partial x}{\partial \zeta}\frac{\partial z}{\partial \xi}\right), \quad Jac_{23} = -\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \xi}\right),$$
$$Jac_{31} = \left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi}\right), \quad Jac_{32} = -\left(\frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial z}{\partial \xi}\right), \quad Jac_{33} = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right),$$

The value of $J \cdot Jac_{ij}$ is denoted as $Jacodet_{ij}$.

Referring to eqn. B.3, we define:

$$A_{11} = \frac{\partial}{\partial \xi} \left(JF \frac{\partial \xi}{\partial x} \right), \qquad A_{12} = \frac{\partial}{\partial \xi} \left(JG \frac{\partial \xi}{\partial y} \right), \qquad A_{13} = \frac{\partial}{\partial \xi} \left(JH \frac{\partial \xi}{\partial z} \right),$$

$$A_{21} = \frac{\partial}{\partial \eta} \left(JF \frac{\partial \eta}{\partial x} \right), \qquad A_{22} = \frac{\partial}{\partial \eta} \left(JG \frac{\partial \eta}{\partial y} \right), \qquad A_{23} = \frac{\partial}{\partial \eta} \left(JH \frac{\partial \eta}{\partial z} \right),$$

$$A_{31} = \frac{\partial}{\partial \zeta} \left(JF \frac{\partial \zeta}{\partial x} \right), \qquad A_{32} = \frac{\partial}{\partial \zeta} \left(JG \frac{\partial \zeta}{\partial y} \right), \qquad A_{33} = \frac{\partial}{\partial \zeta} \left(JH \frac{\partial \zeta}{\partial z} \right).$$

For heat conductive terms, we change A_{ij} with E_{ij} .

The terms:

$$\frac{\partial}{\partial \xi_k} \left(J \frac{\partial \xi_i}{\partial x_j} \right),\,$$

where i = 1, 2, 3 give $\xi_i = \xi, \eta, \zeta$ and $x_i = x, y, z$, are denoted as res_{ijk} in the code. For example:

$$res_{132} = \frac{\partial}{\partial \eta} \left(J \frac{\partial \xi}{\partial z} \right).$$

C.2 Inviscid (Convective) Terms in Curvilinear Coordinates

In the code the inviscid terms are developed in this way:

ξ -derivative

continuity

$$\frac{\partial}{\partial \xi} \left[(\rho u) J \frac{\partial \xi}{\partial x} \right] + \frac{\partial}{\partial \xi} \left[(\rho v) J \frac{\partial \xi}{\partial y} \right] + \frac{\partial}{\partial \xi} \left[(\rho w) J \frac{\partial \xi}{\partial z} \right] = \frac{\partial}{\partial \xi} \left(\rho * temp \right),$$

x-momentum

$$\begin{split} \frac{\partial}{\partial \xi} \left[\left(\rho u u + p \right) J \frac{\partial \xi}{\partial x} \right] + \frac{\partial}{\partial \xi} \left[\left(\rho u v \right) J \frac{\partial \xi}{\partial y} \right] + \frac{\partial}{\partial \xi} \left[\left(\rho u w \right) J \frac{\partial \xi}{\partial z} \right] = \\ \frac{\partial}{\partial \xi} \left(\rho u * temp + p J \frac{\partial \xi}{\partial x} \right), \end{split}$$

y-momentum

$$\begin{split} \frac{\partial}{\partial \xi} \left[\left(\rho v u \right) J \frac{\partial \xi}{\partial x} \right] + \frac{\partial}{\partial \xi} \left[\left(\rho v v + p \right) J \frac{\partial \xi}{\partial y} \right] + \frac{\partial}{\partial \xi} \left[\left(\rho v w \right) J \frac{\partial \xi}{\partial z} \right] = \\ \frac{\partial}{\partial \xi} \left(\rho v * temp + p J \frac{\partial \xi}{\partial y} \right), \end{split}$$

z-momentum

$$\begin{split} \frac{\partial}{\partial \xi} \left[\left(\rho w u \right) J \frac{\partial \xi}{\partial x} \right] + \frac{\partial}{\partial \xi} \left[\left(\rho w v \right) J \frac{\partial \xi}{\partial y} \right] + \frac{\partial}{\partial \xi} \left[\left(\rho w w + p \right) J \frac{\partial \xi}{\partial z} \right] = \\ \frac{\partial}{\partial \xi} \left(\rho w * temp + p J \frac{\partial \xi}{\partial z} \right), \end{split}$$

energy

$$\frac{\partial}{\partial\xi} \left[(E_{tot}u + pu) J \frac{\partial\xi}{\partial x} \right] + \frac{\partial}{\partial\xi} \left[(E_{tot}v + pv) J \frac{\partial\xi}{\partial y} \right] + \frac{\partial}{\partial\xi} \left[(E_{tot}w + pw) J \frac{\partial\xi}{\partial z} \right] = \frac{\partial}{\partial\xi} \left[(E_{tot} + p) * temp \right]$$

where

$$temp = \left(uJ\frac{\partial\xi}{\partial x} + vJ\frac{\partial\xi}{\partial y} + wJ\frac{\partial\xi}{\partial z} \right),$$

η -derivative

continuity

$$\frac{\partial}{\partial \eta} \left[(\rho u) J \frac{\partial \eta}{\partial x} \right] + \frac{\partial}{\partial \eta} \left[(\rho v) J \frac{\partial \eta}{\partial y} \right] + \frac{\partial}{\partial \eta} \left[(\rho w) J \frac{\partial \eta}{\partial z} \right] = \frac{\partial}{\partial \eta} \left(\rho * temp \right),$$

x-momentum

$$\begin{split} \frac{\partial}{\partial \eta} \left[\left(\rho u u + p \right) J \frac{\partial \eta}{\partial x} \right] + \frac{\partial}{\partial \eta} \left[\left(\rho u v \right) J \frac{\partial \eta}{\partial y} \right] + \frac{\partial}{\partial \eta} \left[\left(\rho u w \right) J \frac{\partial \eta}{\partial z} \right] = \\ \frac{\partial}{\partial \eta} \left(\rho u * temp + p J \frac{\partial \eta}{\partial x} \right), \end{split}$$

y-momentum

$$\begin{split} \frac{\partial}{\partial \eta} \left[\left(\rho v u \right) J \frac{\partial \eta}{\partial x} \right] + \frac{\partial}{\partial \eta} \left[\left(\rho v v + p \right) J \frac{\partial \eta}{\partial y} \right] + \frac{\partial}{\partial \eta} \left[\left(\rho v w \right) J \frac{\partial \eta}{\partial z} \right] = \\ \frac{\partial}{\partial \eta} \left(\rho v * temp + p J \frac{\partial \eta}{\partial y} \right), \end{split}$$

z-momentum

$$\begin{split} \frac{\partial}{\partial \eta} \left[\left(\rho w u \right) J \frac{\partial \eta}{\partial x} \right] + \frac{\partial}{\partial \eta} \left[\left(\rho w v \right) J \frac{\partial \eta}{\partial y} \right] + \frac{\partial}{\partial \eta} \left[\left(\rho w w + p \right) J \frac{\partial \eta}{\partial z} \right] = \\ \frac{\partial}{\partial \eta} \left(\rho w * temp + p J \frac{\partial \eta}{\partial z} \right), \end{split}$$

energy

$$\begin{split} \frac{\partial}{\partial \eta} \left[\left(E_{tot}u + pu \right) J \frac{\partial \eta}{\partial x} \right] + \frac{\partial}{\partial \eta} \left[\left(E_{tot}v + pv \right) J \frac{\partial \eta}{\partial y} \right] + \frac{\partial}{\partial \eta} \left[\left(E_{tot}w + pw \right) J \frac{\partial \eta}{\partial z} \right] = \\ \frac{\partial}{\partial \eta} \left[\left(E_{tot} + p \right) * temp \right] \end{split}$$

Appendix C. Nomenclature used in the Full 3D Version of the SBLI Code

where

$$temp = \left(uJ \frac{\partial \eta}{\partial x} + vJ \frac{\partial \eta}{\partial y} + wJ \frac{\partial \eta}{\partial z} \right),$$

ζ -derivative

continuity

$$\frac{\partial}{\partial \zeta} \left[(\rho u) J \frac{\partial \zeta}{\partial x} \right] + \frac{\partial}{\partial \zeta} \left[(\rho u) J \frac{\partial \zeta}{\partial y} \right] + \frac{\partial}{\partial \zeta} \left[(\rho u) J \frac{\partial \zeta}{\partial z} \right] = \frac{\partial}{\partial \zeta} \left(\rho * temp \right),$$

x-momentum

$$\begin{split} \frac{\partial}{\partial \zeta} \left[\left(\rho u u + p \right) J \frac{\partial \zeta}{\partial x} \right] + \frac{\partial}{\partial \zeta} \left[\left(\rho u v \right) J \frac{\partial \zeta}{\partial y} \right] + \frac{\partial}{\partial \zeta} \left[\left(\rho u w \right) J \frac{\partial \zeta}{\partial z} \right] = \\ \frac{\partial}{\partial \zeta} \left(\rho u * temp + p J \frac{\partial \zeta}{\partial x} \right), \end{split}$$

y-momentum

$$\begin{split} \frac{\partial}{\partial \zeta} \left[\left(\rho v u \right) J \frac{\partial \zeta}{\partial x} \right] &+ \frac{\partial}{\partial \zeta} \left[\left(\rho v v + p \right) J \frac{\partial \zeta}{\partial y} \right] + \frac{\partial}{\partial \zeta} \left[\left(\rho v w \right) J \frac{\partial \zeta}{\partial z} \right] = \\ & \frac{\partial}{\partial \zeta} \left(\rho v * temp + p J \frac{\partial \zeta}{\partial y} \right), \end{split}$$

z-momentum

$$\begin{split} \frac{\partial}{\partial \zeta} \left[\left(\rho w u \right) J \frac{\partial \zeta}{\partial x} \right] + \frac{\partial}{\partial \zeta} \left[\left(\rho w v \right) J \frac{\partial \zeta}{\partial y} \right] + \frac{\partial}{\partial \zeta} \left[\left(\rho w w + p \right) J \frac{\partial \zeta}{\partial z} \right] = \\ \frac{\partial}{\partial \zeta} \left(\rho w * temp + p J \frac{\partial \zeta}{\partial z} \right), \end{split}$$

energy

$$\begin{split} \frac{\partial}{\partial \zeta} \left[\left(E_{tot}u + pu \right) J \frac{\partial \zeta}{\partial x} \right] + \frac{\partial}{\partial \zeta} \left[\left(E_{tot}v + pv \right) J \frac{\partial \zeta}{\partial y} \right] + \frac{\partial}{\partial \zeta} \left[\left(E_{tot}w + pw \right) J \frac{\partial \zeta}{\partial z} \right] = \\ \frac{\partial}{\partial \zeta} \left[\left(E_{tot} + p \right) * temp \right], \end{split}$$

Appendix C. Nomenclature used in the Full 3D Version of the SBLI Code

where

$$temp = \left(uJ\frac{\partial\zeta}{\partial x} + vJ\frac{\partial\zeta}{\partial y} + wJ\frac{\partial\zeta}{\partial z} \right).$$

However, this form has been successively changed in the more conservative form for fully 3D curvilinear meshes as explained in section 4.1.

C.3 Viscous (Diffusion) Terms of x-Momentum Equation

We calculate each viscous term of NS equation of x-momentum equation

x-Momentum terms

$$A_{11} = \frac{\partial}{\partial \xi} \left(\tau_{xx} J \frac{\partial \xi}{\partial x} \right) = \frac{\partial \tau_{xx}}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) + \tau_{xx} \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] \left(J \frac{\partial \xi}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right).$$

By defining a new parameter μ' as

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{xx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{2}{3} \frac{\partial u}{\partial y} + \frac{2}{3} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{2}{3} \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \right]}_{A1} + \mu' \frac{\partial u}{\partial y} \Bigg]}_{A1} + \mu' \frac{\partial u}{\partial$$

Appendix C. Nomenclature used in the Full 3D Version of the SBLI Code

$$\begin{split} A1 &= \frac{4}{3} \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) - \frac{2}{3} \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right]. \\ \frac{\partial A1}{\partial \xi} &= \frac{4}{3} \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right] \\ &- \frac{2}{3} \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial y} \right) \right] \\ &- \frac{2}{3} \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial z} \right) \right]. \end{split}$$

$$A_{12} = \frac{\partial}{\partial \xi} \left(J \tau_{xy} \frac{\partial \xi}{\partial y} \right) = \frac{\partial \tau_{xy}}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \tau_{xy} \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \left(J \frac{\partial \xi}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \left[\frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \left[\frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \left[\frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left[J \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left[J \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left[J \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left[J \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left($$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{xy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A_1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A_1} \cdot A_1 + \underbrace{\left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right]}_{A_1} \cdot A_1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}}_{A_1} \cdot A_1 + \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}}_{A_1} \cdot A_1 + \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}}_{A_1} \cdot A_1 + \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}}_{A_1} \cdot A_1 + \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right]}_{A_1} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} \right]_{A_1} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} \right]_{A_1} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} \frac{\partial v}{\partial z}$$

 $\frac{\partial A1}{\partial \xi} = \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{13} = \frac{\partial}{\partial \xi} \left(J\tau_{xz} \frac{\partial \xi}{\partial z} \right) = \frac{\partial \tau_{xz}}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) + \tau_{xz} \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \left(J\frac{\partial \xi}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1=\tau_{xz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} \cdot A1 = \begin{bmatrix} \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \end{bmatrix}$$

$$\frac{\partial A1}{\partial \xi} = \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial z} \right) \right].$$

$$A_{21} = \frac{\partial}{\partial \eta} \left(J\tau_{xx} \frac{\partial \eta}{\partial x} \right) = \frac{\partial \tau_{xx}}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \tau_{xx} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial \tau_{xx}}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] \left(J \frac{\partial \eta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1=\tau_{xx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} \cdot A_{AA}$$

$$A1 = \frac{4}{3} \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) - \frac{2}{3} \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right] \cdot \frac{\partial A1}{\partial \eta} = \frac{4}{3} \left[\frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 u}{\partial \zeta} \frac{\partial \eta}{\partial \eta} \left(\frac{\partial \eta}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial x} \right) \right] - \frac{2}{3} \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right] - \frac{2}{3} \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right]$$

$$A_{22} = \frac{\partial}{\partial \eta} \left(J \tau_{xy} \frac{\partial \eta}{\partial y} \right) = \frac{\partial \tau_{xy}}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \tau_{xy} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) = \frac{\partial}{\partial \eta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \left(J \frac{\partial \eta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{xy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} \cdot A_{A1}$$
$$A1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix} \cdot A_{A1}$$

 $\frac{\partial A1}{\partial \eta} = \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{23} = \frac{\partial}{\partial \eta} \left(J \tau_{xz} \frac{\partial \eta}{\partial z} \right) = \frac{\partial \tau_{xz}}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) + \tau_{xz} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) = \frac{\partial}{\partial \eta} \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \left(J \frac{\partial \eta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right).$$

If we indicate with

$$\mu'=\frac{\mu}{Re},$$

and

$$A1 = \tau_{xz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} \cdot A1$$
$$A1 = \begin{bmatrix} \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \end{bmatrix}.$$

 $\frac{\partial A1}{\partial \eta} = \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 w}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{31} = \frac{\partial}{\partial \zeta} \left(J\tau_{xx} \frac{\partial \zeta}{\partial x} \right) = \frac{\partial \tau_{xx}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) + \tau_{xx} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) = \frac{\partial \tau_{xx}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] \left(J \frac{\partial \zeta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1=\tau_{xx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A_1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A_1}.$$

$$A_1 = \frac{4}{3} \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) - \frac{2}{3} \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \end{bmatrix}$$

$$\frac{\partial A_1}{\partial \zeta} = \frac{4}{3} \begin{bmatrix} \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 u}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right]$$

$$- \frac{2}{3} \begin{bmatrix} \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right]$$

$$- \frac{2}{3} \begin{bmatrix} \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$$

$$A_{32} = \frac{\partial}{\partial \zeta} \left(J \tau_{xy} \frac{\partial \zeta}{\partial y} \right) = \frac{\partial \tau_{xy}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) + \tau_{xy} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \left(J \frac{\partial \zeta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{xy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} \cdot A1 = - \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right].$$

 $\frac{\partial A1}{\partial \zeta} = \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{33} = \frac{\partial}{\partial \zeta} \left(J \tau_{xz} \frac{\partial \zeta}{\partial z} \right) = \frac{\partial \tau_{xz}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \tau_{xz} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \left(J \frac{\partial \zeta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \left[\frac{\partial}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right] \right] \frac{\partial}{\partial \zeta} \left[\frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right]$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{xz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} \cdot A1 = \begin{bmatrix} \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \end{bmatrix}.$$

 $\frac{\partial A1}{\partial \zeta} = \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 w}{\partial \zeta^2} \frac{\partial \zeta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \right) \right].$

Appendix C. Nomenclature used in the Full 3D Version of the SBLI Code

C.4 Viscous (Diffusion) Terms of y-Momentum Equation

We rewrite each viscous term of NS equation of y-momentum equation

y-Momentum terms

$$A_{11} = \frac{\partial}{\partial \xi} \left(J \tau_{yx} \frac{\partial \xi}{\partial x} \right) = \frac{\partial \tau_{yx}}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) + \tau_{yx} \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \left(J \frac{\partial \xi}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

 $A1 = \tau_{yx}/\mu',$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} \right]}_{A1}.$$
$$A1 = \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right].$$

 $\frac{\partial A1}{\partial \xi} = \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{12} = \frac{\partial}{\partial \xi} \left(J \tau_{yy} \frac{\partial \xi}{\partial y} \right) = \frac{\partial \tau_{yy}}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \tau_{yy} \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right] \left(J \frac{\partial \xi}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

2

$$A1 = \tau_{yy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A_1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A_1}.$$

$$A_1 = \frac{4}{3} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) - \frac{2}{3} \left[\left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right].$$

$$\frac{\partial A_1}{\partial \xi} = \frac{4}{3} \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial y} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial y} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial \xi} \left(\frac{\partial \zeta}{\partial y} \right) \right] \\ - \frac{2}{3} \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial x} \right) \right] .$$

$$A_{13} = \frac{\partial}{\partial \xi} \left(J \tau_{yz} \frac{\partial \xi}{\partial z} \right) = \frac{\partial \tau_{yz}}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right) + \tau_{yz} \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \left(J \frac{\partial \xi}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{yz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} \cdot \\ A1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}.$$

 $\frac{\partial A1}{\partial \xi} = \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{21} = \frac{\partial}{\partial \eta} \left(J \tau_{yx} \frac{\partial \eta}{\partial x} \right) = \frac{\partial \tau_{yx}}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \tau_{yx} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial}{\partial \eta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \left(J \frac{\partial \eta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left[\frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left($$

If we indicate with

$$\mu'=\frac{\mu}{Re},$$

and

$$A1 = \tau_{yx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} \cdot A1 = \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right].$$

$$\begin{split} \frac{\partial A1}{\partial \eta} &= \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ &+ \left[\frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right]. \end{split}$$

$$A_{22} = \frac{\partial}{\partial \eta} \left(J\tau_{yy} \frac{\partial \eta}{\partial y} \right) = \frac{\partial \tau_{yy}}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \tau_{yy} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) = \frac{\partial \tau_{yy}}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right] \left(J \frac{\partial \eta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{yy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A1} \cdot A_{AA}$$

$$A1 = \frac{4}{3} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) - \frac{2}{3} \left[\left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right] \cdot A_{AA}$$

$$\frac{\partial A1}{\partial \eta} = \frac{4}{3} \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial \eta} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial \eta} \left(\frac{\partial v}{\partial y} \right) + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right] - \frac{2}{3} \left[\frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial^2 w}{\partial \eta^2} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial \eta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial \eta} \left(\frac{\partial \zeta}{\partial x} \right) \right] - \frac{2}{3} \left[\frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial^2 w}{\partial \eta^2} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial \eta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \eta}{\partial \eta} \left(\frac{\partial \zeta}{\partial z} \right) \right].$$

$$A_{23} = \frac{\partial}{\partial \eta} \left(J \tau_{yz} \frac{\partial \eta}{\partial z} \right) = \frac{\partial \tau_{yz}}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) + \tau_{yz} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) = \frac{\partial}{\partial \eta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \left(J \frac{\partial \eta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{yz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} \cdot \\ A1 = \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right].$$

 $\frac{\partial A1}{\partial \eta} = \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 w}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$
$$A_{31} = \frac{\partial}{\partial \zeta} \left(J \tau_{yx} \frac{\partial \zeta}{\partial x} \right) = \frac{\partial \tau_{yx}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) + \tau_{yx} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \left(J \frac{\partial \zeta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{yx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right]}_{A1}.$$
$$A1 = \left[\left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right]$$

 $\frac{\partial A1}{\partial \zeta} = \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{32} = \frac{\partial}{\partial \zeta} \left(J \tau_{yy} \frac{\partial \zeta}{\partial y} \right) = \frac{\partial \tau_{yy}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) + \tau_{yy} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) = \frac{\partial \tau_{yy}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right] \left(J \frac{\partial \zeta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{yy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A_1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right) \right]}_{A_1}.$$

$$A_1 = \frac{4}{3} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) - \frac{2}{3} \left[\left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right].$$

$$\frac{\partial A_1}{\partial \zeta} = \frac{4}{3} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right].$$

$$\frac{\partial A_1}{\partial \zeta} = \frac{4}{3} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$$

$$\frac{\partial A_1}{\partial \zeta} = \frac{4}{3} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$$

$$A_{33} = \frac{\partial}{\partial \zeta} \left(J\tau_{yz} \frac{\partial \zeta}{\partial z} \right) = \frac{\partial \tau_{yz}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \tau_{yz} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \left(J \frac{\partial \zeta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{yz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} \cdot \\ A1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}.$$

 $\frac{\partial A1}{\partial \zeta} = \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 w}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

C.5 Viscous (Diffusion) Terms of z-Momentum Equation

We rewrite each viscous term of NS equation of z-momentum equation

z-Momentum terms

$$A_{11} = \frac{\partial}{\partial \xi} \left(J\tau_{zx} \frac{\partial \xi}{\partial x} \right) = \frac{\partial \tau_{zx}}{\partial \xi} \left(J\frac{\partial \xi}{\partial x} \right) + \tau_{zx} \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial x} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \left(J\frac{\partial \xi}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial x} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

 $A1 = \tau_{zx}/\mu',$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1}.$$
$$A1 = \left[\left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \right].$$

 $\frac{\partial A1}{\partial \xi} = \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial z} \right) \right]$

$$A_{12} = \frac{\partial}{\partial \xi} \left(J \tau_{zy} \frac{\partial \xi}{\partial y} \right) = \frac{\partial \tau_{zy}}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \tau_{zy} \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \left(J \frac{\partial \xi}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) = \frac{\partial}{\partial \xi} \left[\frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial y} \right] \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial}{\partial \xi} \right$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{zy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} \cdot \\ A1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}$$

 $\frac{\partial A1}{\partial \xi} = \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \xi} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{13} = \frac{\partial}{\partial \xi} \left(J\tau_{zz} \frac{\partial \xi}{\partial z} \right) = \frac{\partial \tau_{zz}}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) + \tau_{zz} \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) = \frac{\partial}{\partial \xi} \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right] \left(J\frac{\partial \xi}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right] \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \xi} \left(J\frac{\partial \xi}{\partial z} \right) = \frac{A}{A}$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{zz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \xi} \underbrace{\left[\left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right]}_{A1} \cdot A_{AA}$$

$$A1 = \frac{4}{3} \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) - \frac{2}{3} \left[\left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right] \cdot \frac{\partial A1}{\partial \xi} = \frac{4}{3} \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \xi} \frac{\partial (\xi)}{\partial \xi} + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \frac{\partial (\xi)}{\partial z} + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] - \frac{2}{3} \left[\frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \xi} \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial (\psi)}{\partial \eta} \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\psi)}{\partial z} \right) \frac{\partial (\xi)}{\partial z} + \frac{\partial (\psi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] - \frac{2}{3} \left[\frac{\partial^2 v}{\partial \xi^2} \frac{\partial (\xi)}{\partial \xi} + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial y} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial y} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial y} \right) \right] \cdot \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial y} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] \cdot \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] \cdot \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial y} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] \cdot \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial y} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) \right] \cdot \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)}{\partial z} \right) + \frac{\partial (\xi)}{\partial \xi} \left(\frac{\partial (\xi)$$

$$A_{21} = \frac{\partial}{\partial \eta} \left(J \tau_{zx} \frac{\partial \eta}{\partial x} \right) = \frac{\partial \tau_{zx}}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + \tau_{zx} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) = \frac{\partial}{\partial \eta} \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \left(J \frac{\partial \eta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{zx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} \cdot \frac{\partial u}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right]}_{A1} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right]}_{A1} \cdot \frac{\partial A1}{\partial \eta} = \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \xi} \frac{\partial^2 \xi}{\partial x \partial \eta} + \frac{\partial^2 w}{\partial \eta \partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial^2 \eta}{\partial x \partial \eta} + \frac{\partial^2 w}{\partial \zeta \partial \eta} \frac{\partial \zeta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial^2 \zeta}{\partial x \partial \eta} \right) \\ + \left(\frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial z \partial \eta} + \frac{\partial^2 u}{\partial \eta \partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial z \partial \eta} + \frac{\partial^2 u}{\partial \zeta \partial \eta} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial^2 \zeta}{\partial z \partial \eta} \right)$$

$$A_{22} = \frac{\partial}{\partial \eta} \left(J \tau_{zy} \frac{\partial \eta}{\partial y} \right) = \frac{\partial \tau_{zy}}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + \tau_{zy} \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) = \frac{\partial}{\partial \eta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \left(J \frac{\partial \eta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right)$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{zy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \eta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} \cdot A1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \end{bmatrix}.$$

 $\frac{\partial A1}{\partial \eta} = \left[\frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial^2 v}{\partial \eta^2} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial v}{\partial \zeta} \right) \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \eta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial^2 w}{\partial \eta^2} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \eta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial w}{\partial \zeta} \right) \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \eta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$A_{23} = \frac{\partial}{\partial \eta} \left(J\tau_{zz} \frac{\partial \eta}{\partial z} \right) = \frac{\partial \tau_{zz}}{\partial \eta} \left(J\frac{\partial \eta}{\partial z} \right) + \tau_{zz} \frac{\partial}{\partial \eta} \left(J\frac{\partial \eta}{\partial z} \right) = \frac{\partial \tau_{zz}}{\partial \eta} \left(J\frac{\partial \eta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right] \left(J\frac{\partial \eta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right] \frac{\partial}{\partial \eta} \left(J\frac{\partial \eta}{\partial z} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1=\tau_{zz}/\mu',$$

we have

$$A_{31} = \frac{\partial}{\partial \zeta} \left(J \tau_{zx} \frac{\partial \zeta}{\partial x} \right) = \frac{\partial \tau_{zx}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) + \tau_{zx} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \left(J \frac{\partial \zeta}{\partial x} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right),$$

If we indicate with

$$\mu'=\frac{\mu}{Re},$$

and

$$A1=\tau_{zx}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right]}_{A1} \cdot A1$$
$$A1 = \begin{bmatrix} \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \end{bmatrix}.$$

 $\frac{\partial A1}{\partial \zeta} = \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 w}{\partial \zeta^2} \frac{\partial \zeta}{\partial x} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\ + \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \right) \right].$

$$A_{32} = \frac{\partial}{\partial \zeta} \left(J \tau_{zy} \frac{\partial \zeta}{\partial y} \right) = \frac{\partial \tau_{zy}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) + \tau_{zy} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) = \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \left(J \frac{\partial \zeta}{\partial y} \right) + \left[\frac{\mu}{Re} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right).$$

If we indicate with

$$\mu' = \frac{\mu}{Re},$$

and

$$A1 = \tau_{zy}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right]}_{A1} \cdot A1 = \begin{bmatrix} \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right]$$

•

 $\frac{\partial A1}{\partial \zeta} = \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\ + \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 w}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right].$

$$\begin{split} A_{33} &= \frac{\partial}{\partial \zeta} \left(J \tau_{zz} \frac{\partial \zeta}{\partial z} \right) = \\ & \frac{\partial \tau_{zz}}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + \tau_{zz} \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = \\ & \frac{\partial}{\partial \zeta} \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right] \left(J \frac{\partial \zeta}{\partial z} \right) + \left[\frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right] \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right). \end{split}$$
If we indicate with

 $\mu' = \frac{\mu}{Re},$

and

$$A1=\tau_{zz}/\mu',$$

we have

$$A = \frac{\partial \mu'}{\partial \zeta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right]}_{A_1} + \mu' \frac{\partial}{\partial \zeta} \underbrace{\left[\left(\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \right]}_{A_1}.$$

$$A_1 = \frac{4}{3} \left(\frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) - \frac{2}{3} \left[\left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \left(\frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) \right].$$

$$\frac{\partial A_1}{\partial \zeta} = \frac{4}{3} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial w}{\partial \eta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 w}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial w}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \right) \right] \\- \frac{2}{3} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right] \\- \frac{2}{3} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial v}{\partial \eta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 v}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right]$$

C.6 Heat Conductivity Terms in the Energy Equations

We rewrite each term of heat conductivity in energy equation

conductivity energy terms

$$E_{11} = \frac{\partial}{\partial \xi} \left(q_x J \frac{\partial \xi}{\partial x} \right) =$$
$$\frac{\partial q_x}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) + q_x \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right) =$$
$$\frac{\partial}{\partial \xi} \left(\mu'' \frac{\partial T}{\partial x} \right) \left(J \frac{\partial \xi}{\partial x} \right) + \left(\mu'' \frac{\partial T}{\partial x} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial x} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right),$$

$$\frac{\partial}{\partial\xi} \left(\mu'' \frac{\partial T}{\partial x} \right) = \frac{\partial\mu''}{\partial\xi} \left(\frac{\partial T}{\partial\xi} \frac{\partial\xi}{\partial x} + \frac{\partial T}{\partial\eta} \frac{\partial\eta}{\partial x} + \frac{\partial T}{\partial\zeta} \frac{\partial\zeta}{\partial x} \right) + \\ \mu'' \left[\frac{\partial^2 T}{\partial\xi^2} \frac{\partial\xi}{\partial x} + \frac{\partial T}{\partial\xi} \frac{\partial}{\partial\xi} \left(\frac{\partial\xi}{\partial x} \right) + \frac{\partial}{\partial\xi} \left(\frac{\partial T}{\partial\eta} \right) \frac{\partial\eta}{\partial x} + \frac{\partial T}{\partial\eta} \frac{\partial}{\partial\xi} \left(\frac{\partial\eta}{\partial x} \right) + \frac{\partial}{\partial\xi} \left(\frac{\partial T}{\partial\zeta} \right) \frac{\partial\zeta}{\partial x} + \frac{\partial T}{\partial\zeta} \frac{\partial}{\partial\xi} \left(\frac{\partial\zeta}{\partial x} \right) \right]$$

$$E_{12} = \frac{\partial}{\partial \xi} \left(q_y J \frac{\partial \xi}{\partial y} \right) =$$
$$\frac{\partial q_y}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) + q_y \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right) =$$
$$\frac{\partial}{\partial \xi} \left(\mu'' \frac{\partial T}{\partial y} \right) \left(J \frac{\partial \xi}{\partial y} \right) + \left(\mu'' \frac{\partial T}{\partial y} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial y} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial y} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right),\,$$

$$\frac{\partial}{\partial\xi} \left(\mu'' \frac{\partial T}{\partial y} \right) = \frac{\partial\mu''}{\partial\xi} \left(\frac{\partial T}{\partial\xi} \frac{\partial\xi}{\partial y} + \frac{\partial T}{\partial\eta} \frac{\partial\eta}{\partial y} + \frac{\partial T}{\partial\zeta} \frac{\partial\zeta}{\partial y} \right) + \\ \mu'' \left[\frac{\partial^2 T}{\partial\xi^2} \frac{\partial\xi}{\partial y} + \frac{\partial T}{\partial\xi} \frac{\partial}{\partial\xi} \left(\frac{\partial\xi}{\partial y} \right) + \frac{\partial}{\partial\xi} \left(\frac{\partial T}{\partial\eta} \right) \frac{\partial\eta}{\partial y} + \frac{\partial T}{\partial\eta} \frac{\partial}{\partial\xi} \left(\frac{\partial\eta}{\partial y} \right) + \frac{\partial}{\partial\xi} \left(\frac{\partial T}{\partial\zeta} \right) \frac{\partial\zeta}{\partial y} + \frac{\partial T}{\partial\zeta} \frac{\partial}{\partial\xi} \left(\frac{\partial\zeta}{\partial y} \right) \right]$$

$$E_{13} = \frac{\partial}{\partial \xi} \left(q_z J \frac{\partial \xi}{\partial z} \right) = \frac{\partial q_z}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right) + q_z \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right) = \frac{\partial}{\partial \xi} \left(\mu'' \frac{\partial T}{\partial z} \right) \left(J \frac{\partial \xi}{\partial z} \right) + \left(\mu'' \frac{\partial T}{\partial z} \right) \frac{\partial}{\partial \xi} \left(J \frac{\partial \xi}{\partial z} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial z} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right),\,$$

$$\frac{\partial}{\partial\xi} \left(\mu'' \frac{\partial T}{\partial z} \right) = \frac{\partial\mu''}{\partial\xi} \left(\frac{\partial T}{\partial\xi} \frac{\partial\xi}{\partial z} + \frac{\partial T}{\partial\eta} \frac{\partial\eta}{\partial z} + \frac{\partial T}{\partial\zeta} \frac{\partial\zeta}{\partial z} \right) + \\ \mu'' \left[\frac{\partial^2 T}{\partial\xi^2} \frac{\partial\xi}{\partial z} + \frac{\partial T}{\partial\xi} \frac{\partial}{\partial\xi} \left(\frac{\partial\xi}{\partial z} \right) + \frac{\partial}{\partial\xi} \left(\frac{\partial T}{\partial\eta} \right) \frac{\partial\eta}{\partial z} + \frac{\partial T}{\partial\eta} \frac{\partial}{\partial\xi} \left(\frac{\partial\eta}{\partial z} \right) + \frac{\partial}{\partial\xi} \left(\frac{\partial T}{\partial\zeta} \right) \frac{\partial\zeta}{\partial z} + \frac{\partial T}{\partial\zeta} \frac{\partial}{\partial\xi} \left(\frac{\partial\zeta}{\partial z} \right) \right]$$

$$E_{21} = \frac{\partial}{\partial \eta} \left(q_x J \frac{\partial \eta}{\partial x} \right) =$$
$$\frac{\partial q_x}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) + q_x \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right) =$$
$$\frac{\partial}{\partial \eta} \left(\mu'' \frac{\partial T}{\partial x} \right) \left(J \frac{\partial \eta}{\partial x} \right) + \left(\mu'' \frac{\partial T}{\partial x} \right) \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial x} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right),$$

$$\frac{\partial}{\partial\eta} \left(\mu'' \frac{\partial T}{\partial x} \right) = \frac{\partial\mu''}{\partial\eta} \left(\frac{\partial T}{\partial\xi} \frac{\partial\xi}{\partial x} + \frac{\partial T}{\partial\eta} \frac{\partial\eta}{\partial x} + \frac{\partial T}{\partial\zeta} \frac{\partial\zeta}{\partial x} \right) + \\ \mu'' \left[\frac{\partial}{\partial\eta} \left(\frac{\partial T}{\partial\xi} \right) \frac{\partial\xi}{\partial x} + \frac{\partial T}{\partial\xi} \frac{\partial}{\partial\eta} \left(\frac{\partial\xi}{\partial x} \right) + \frac{\partial^2 T}{\partial\eta^2} \frac{\partial\eta}{\partial x} + \frac{\partial T}{\partial\eta} \frac{\partial}{\partial\eta} \left(\frac{\partial\eta}{\partial x} \right) + \frac{\partial}{\partial\eta} \left(\frac{\partial T}{\partial\zeta} \right) \frac{\partial\zeta}{\partial x} + \frac{\partial T}{\partial\zeta} \frac{\partial}{\partial\eta} \left(\frac{\partial\zeta}{\partial x} \right) \right]$$

$$E_{22} = \frac{\partial}{\partial \eta} \left(q_y J \frac{\partial \eta}{\partial y} \right) = \frac{\partial q_y}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) + q_y \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right) = \frac{\partial}{\partial \eta} \left(\mu'' \frac{\partial T}{\partial y} \right) \left(J \frac{\partial \eta}{\partial y} \right) + \left(\mu'' \frac{\partial T}{\partial y} \right) \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial y} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right),\,$$

$$\frac{\partial}{\partial\eta} \left(\mu'' \frac{\partial T}{\partial y} \right) = \frac{\partial\mu''}{\partial\eta} \left(\frac{\partial T}{\partial\xi} \frac{\partial\xi}{\partial y} + \frac{\partial T}{\partial\eta} \frac{\partial\eta}{\partial y} + \frac{\partial T}{\partial\zeta} \frac{\partial\zeta}{\partial y} \right) + \\ \mu'' \left[\frac{\partial}{\partial\eta} \left(\frac{\partial T}{\partial\xi} \right) \frac{\partial\xi}{\partial y} + \frac{\partial T}{\partial\xi} \frac{\partial}{\partial\eta} \left(\frac{\partial\xi}{\partial y} \right) + \frac{\partial^2 T}{\partial\eta^2} \frac{\partial\eta}{\partial y} + \frac{\partial T}{\partial\eta} \frac{\partial}{\partial\eta} \left(\frac{\partial\eta}{\partial y} \right) + \frac{\partial}{\partial\eta} \left(\frac{\partial T}{\partial\zeta} \right) \frac{\partial\zeta}{\partial y} + \frac{\partial T}{\partial\zeta} \frac{\partial}{\partial\eta} \left(\frac{\partial\zeta}{\partial y} \right) \right]$$

$$E_{23} = \frac{\partial}{\partial \eta} \left(q_z J \frac{\partial \eta}{\partial z} \right) = \frac{\partial q_z}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) + q_z \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right) = \frac{\partial}{\partial \eta} \left(\mu'' \frac{\partial T}{\partial z} \right) \left(J \frac{\partial \eta}{\partial z} \right) + \left(\mu'' \frac{\partial T}{\partial z} \right) \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial z} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right),$$

$$\frac{\partial}{\partial\eta} \left(\mu'' \frac{\partial T}{\partial z} \right) = \frac{\partial\mu''}{\partial\eta} \left(\frac{\partial T}{\partial\xi} \frac{\partial\xi}{\partial z} + \frac{\partial T}{\partial\eta} \frac{\partial\eta}{\partial z} + \frac{\partial T}{\partial\zeta} \frac{\partial\zeta}{\partial z} \right) + \\ \mu'' \left[\frac{\partial}{\partial\eta} \left(\frac{\partial T}{\partial\xi} \right) \frac{\partial\xi}{\partial z} + \frac{\partial T}{\partial\xi} \frac{\partial}{\partial\eta} \left(\frac{\partial\xi}{\partial z} \right) + \frac{\partial^2 T}{\partial\eta^2} \frac{\partial\eta}{\partial z} + \frac{\partial T}{\partial\eta} \frac{\partial}{\partial\eta} \left(\frac{\partial\eta}{\partial z} \right) + \frac{\partial}{\partial\eta} \left(\frac{\partial T}{\partial\zeta} \right) \frac{\partial\zeta}{\partial z} + \frac{\partial T}{\partial\zeta} \frac{\partial}{\partial\eta} \left(\frac{\partial\zeta}{\partial z} \right) \right]$$

$$E_{31} = \frac{\partial}{\partial \zeta} \left(q_x J \frac{\partial \zeta}{\partial x} \right) = \frac{\partial q_x}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) + q_x \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right) = \frac{\partial}{\partial \zeta} \left(\mu'' \frac{\partial T}{\partial x} \right) \left(J \frac{\partial \zeta}{\partial x} \right) + \left(\mu'' \frac{\partial T}{\partial x} \right) \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial x} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial x} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right)$$

$$\frac{\partial}{\partial \zeta} \left(\mu'' \frac{\partial T}{\partial x} \right) = \frac{\partial \mu''}{\partial \zeta} \left(\frac{\partial T}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial T}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right) + \\ \mu'' \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial T}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial T}{\partial \zeta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial T}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 T}{\partial \zeta^2} \frac{\partial \zeta}{\partial x} + \frac{\partial T}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial x} \right) \right] +$$

$$E_{32} = \frac{\partial}{\partial \zeta} \left(q_y J \frac{\partial \zeta}{\partial y} \right) = \frac{\partial q_y}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) + q_y \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right) = \frac{\partial}{\partial \zeta} \left(\mu'' \frac{\partial T}{\partial y} \right) \left(J \frac{\partial \zeta}{\partial y} \right) + \left(\mu'' \frac{\partial T}{\partial y} \right) \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial y} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right)$$

$$\frac{\partial}{\partial \zeta} \left(\mu'' \frac{\partial T}{\partial y} \right) = \frac{\partial \mu''}{\partial \zeta} \left(\frac{\partial T}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial T}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right) + \\ \mu'' \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial T}{\partial \xi} \right) \frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial y} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial T}{\partial \zeta} \right) \frac{\partial \eta}{\partial y} + \frac{\partial T}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial^2 T}{\partial \zeta^2} \frac{\partial \zeta}{\partial y} + \frac{\partial T}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial y} \right) \right]$$

$$E_{33} = \frac{\partial}{\partial \zeta} \left(q_z J \frac{\partial \zeta}{\partial z} \right) = \frac{\partial q_z}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) + q_z \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right) = \frac{\partial}{\partial \zeta} \left(\mu'' \frac{\partial T}{\partial z} \right) \left(J \frac{\partial \zeta}{\partial z} \right) + \left(\mu'' \frac{\partial T}{\partial z} \right) \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right),$$

where:

$$\mu'' = \frac{\mu}{RePrM^2(\gamma - 1)}.$$

Then, we have:

$$\mu''\frac{\partial T}{\partial z} = \mu''\left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial T}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)$$

$$\frac{\partial}{\partial \zeta} \left(\mu'' \frac{\partial T}{\partial z} \right) = \frac{\partial \mu''}{\partial \zeta} \left(\frac{\partial T}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial T}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) + \\ \mu'' \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial T}{\partial \xi} \right) \frac{\partial \xi}{\partial z} + \frac{\partial T}{\partial \xi} \frac{\partial}{\partial \zeta} \left(\frac{\partial \xi}{\partial z} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial T}{\partial \zeta} \right) \frac{\partial \eta}{\partial z} + \frac{\partial T}{\partial \eta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \eta}{\partial z} \right) + \frac{\partial^2 T}{\partial \zeta^2} \frac{\partial \zeta}{\partial z} + \frac{\partial T}{\partial \zeta} \frac{\partial}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \right) \right]$$

C.7 Viscous Diffusion Terms of Energy Equation

For the energy equation we have the following ξ -derivative terms

$$\frac{\partial}{\partial\xi} \left[\left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz} \right) J \frac{\partial\xi}{\partialx} \right] = \left(\frac{\partial u}{\partial\xi} \tau_{xx} + u \frac{\partial \tau_{xx}}{\partial\xi} + \frac{\partial v}{\partial\xi} \tau_{xy} + v \frac{\partial \tau_{xy}}{\partial\xi} + \frac{\partial w}{\partial\xi} \tau_{xz} + w \frac{\partial \tau_{xz}}{\partial\xi} \right) J \frac{\partial\xi}{\partialx} + \left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz} \right) \frac{\partial}{\partial\xi} \left(J \frac{\partial\xi}{\partialx} \right).$$

We have already rewritten the derivative of the stress tensor terms as presented is C.3, C.4 and C.5.

The other two terms are:

$$\frac{\partial}{\partial\xi} \left[\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \right) J \frac{\partial\xi}{\partial y} \right] = \left(\frac{\partial u}{\partial\xi} \tau_{yx} + u \frac{\partial \tau_{yx}}{\partial\xi} + \frac{\partial v}{\partial\xi} \tau_{yy} + v \frac{\partial \tau_{yy}}{\partial\xi} + \frac{\partial w}{\partial\xi} \tau_{yz} + w \frac{\partial \tau_{yz}}{\partial\xi} \right) J \frac{\partial\xi}{\partial y} + \left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \right) \frac{\partial}{\partial\xi} \left(J \frac{\partial\xi}{\partial y} \right),$$

$$\frac{\partial}{\partial\xi} \left[\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \right) J \frac{\partial\xi}{\partialz} \right] = \left(\frac{\partial u}{\partial\xi} \tau_{zx} + u \frac{\partial \tau_{zx}}{\partial\xi} + \frac{\partial v}{\partial\xi} \tau_{zy} + v \frac{\partial \tau_{zy}}{\partial\xi} + \frac{\partial w}{\partial\xi} \tau_{zz} + w \frac{\partial \tau_{zz}}{\partial\xi} \right) J \frac{\partial\xi}{\partialz} + \left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \right) \frac{\partial}{\partial\xi} \left(J \frac{\partial\xi}{\partialz} \right).$$

For η -derivative we have

$$\frac{\partial}{\partial \eta} \left[\left(u \tau_{xx} + v \tau_{xy} + w \tau_{xz} \right) J \frac{\partial \eta}{\partial x} \right] =$$

 $\left(\frac{\partial u}{\partial \eta}\tau_{xx} + u\frac{\partial \tau_{xx}}{\partial \eta} + \frac{\partial v}{\partial \eta}\tau_{xy} + v\frac{\partial \tau_{xy}}{\partial \eta} + \frac{\partial w}{\partial \eta}\tau_{xz} + w\frac{\partial \tau_{xz}}{\partial \eta}\right)J\frac{\partial \eta}{\partial x} + \left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}\right)\frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial x}\right),$

$$\frac{\partial}{\partial \eta} \left[\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \right) J \frac{\partial \eta}{\partial y} \right] = \\ \left(\frac{\partial u}{\partial \eta} \tau_{yx} + u \frac{\partial \tau_{yx}}{\partial \eta} + \frac{\partial v}{\partial \eta} \tau_{yy} + v \frac{\partial \tau_{yy}}{\partial \eta} + \frac{\partial w}{\partial \eta} \tau_{yz} + w \frac{\partial \tau_{yz}}{\partial \eta} \right) J \frac{\partial \eta}{\partial y} + \left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \right) \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial y} \right),$$

$$\frac{\partial}{\partial \eta} \left[\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \right) J \frac{\partial \eta}{\partial z} \right] = \\ \left(\frac{\partial u}{\partial \eta} \tau_{zx} + u \frac{\partial \tau_{zx}}{\partial \eta} + \frac{\partial v}{\partial \eta} \tau_{zy} + v \frac{\partial \tau_{zy}}{\partial \eta} + \frac{\partial w}{\partial \eta} \tau_{zz} + w \frac{\partial \tau_{zz}}{\partial \eta} \right) J \frac{\partial \eta}{\partial z} + \left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \right) \frac{\partial}{\partial \eta} \left(J \frac{\partial \eta}{\partial z} \right).$$

And for ζ derivative

$$\frac{\partial}{\partial \zeta} \left[\left(u \tau_{xx} + v \tau_{xy} + w \tau_{xz} \right) J \frac{\partial \zeta}{\partial x} \right] =$$

 $\left(\frac{\partial u}{\partial \zeta}\tau_{xx} + u\frac{\partial \tau_{xx}}{\partial \zeta} + \frac{\partial v}{\partial \zeta}\tau_{xy} + v\frac{\partial \tau_{xy}}{\partial \zeta} + \frac{\partial w}{\partial \zeta}\tau_{xz} + w\frac{\partial \tau_{xz}}{\partial \zeta}\right)J\frac{\partial \zeta}{\partial x} + \left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}\right)\frac{\partial}{\partial \zeta}\left(J\frac{\partial \zeta}{\partial x}\right),$

$$\frac{\partial}{\partial \zeta} \left[\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \right) J \frac{\partial \zeta}{\partial y} \right] = \left(\frac{\partial u}{\partial \zeta} \tau_{yx} + u \frac{\partial \tau_{yx}}{\partial \zeta} + \frac{\partial v}{\partial \zeta} \tau_{yy} + v \frac{\partial \tau_{yy}}{\partial \zeta} + \frac{\partial w}{\partial \zeta} \tau_{yz} + w \frac{\partial \tau_{yz}}{\partial \zeta} \right) J \frac{\partial \zeta}{\partial y} + \left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \right) \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial y} \right),$$

$$\frac{\partial}{\partial \zeta} \left[\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \right) J \frac{\partial \zeta}{\partial z} \right] = \left(\frac{\partial u}{\partial \zeta} \tau_{zx} + u \frac{\partial \tau_{zx}}{\partial \zeta} + \frac{\partial v}{\partial \zeta} \tau_{zy} + v \frac{\partial \tau_{zy}}{\partial \zeta} + \frac{\partial w}{\partial \zeta} \tau_{zz} + w \frac{\partial \tau_{zz}}{\partial \zeta} \right) J \frac{\partial \zeta}{\partial z} + \left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \right) \frac{\partial}{\partial \zeta} \left(J \frac{\partial \zeta}{\partial z} \right).$$

Appendix D

Digital Filter Method

Here we describe the digital filter method for generating the inflow conditions applied in Chapter 6. Considering a general point at the inflow plane of the domain (0,y,z). As originally proposed by Lund *et al.* (1998), the single point time correlation can be expressed as:

$$\begin{bmatrix} u(0, y, z, t) \\ u(0, y, z, t) \\ u(0, y, z, t) \end{bmatrix} = \begin{bmatrix} u(0, y, z) \\ u(0, y, z) \\ u(0, y, z) \end{bmatrix} + \begin{bmatrix} \sqrt{R_{11}} & 0 & 0 \\ R_{21}/\sqrt{R_{11}} & \sqrt{R_{22} - (R_{21}/\sqrt{R_{11}})^2} & 0 \\ 0 & 0 & \sqrt{R_{33}} \end{bmatrix} \begin{bmatrix} \rho^u(y, z) \\ \rho^v(y, z) \\ \rho^w(y, z) \end{bmatrix}$$
(D.1)

where R_{ij} , for i, j=1,2,3, are the prescribed Reynolds stresses and ρ^u, ρ^v, ρ^w are the filtered fields which contain the enforced two-point spatial correlation functions as well as the prescribed streamwise correlation that we want to match. The above field is linked to discrete filter operator $F_N(r_k) \equiv v_k$ as following:

$$\rho_k = v_k^{old} e^{\left(\frac{-\pi\Delta t}{2\tau}\right)} + v_k \sqrt{1 - e^{\left(\frac{-\pi\Delta t}{\tau}\right)}},\tag{D.2}$$

where Δt is the time step and τ is the Lagrangian time scale ($\tau = I_x/U$ in the present calculations, where U and I_x are the prescribed inlet mean streamwise velocity profile and integral length scale, respectively).

The following table gives the parameter values used for generalizing the inflow

condition:

velocity component	u	v	w
I_x in δ^* units	10	4	4
$N_{F_y} = 2I_y / \Delta y$	$35^{a}-65^{b}$	$45^a - 85^b$	$30^a - 40^b$
$N_{F_z} = 2I_z/\Delta z$	15	15	30

^a $y \leq y_{lim}$, where $y_{lim} = 1 \delta^*$

^b $y > y_{lim}$, where $y_{lim}=1 \ \delta^*$

where I_y and I_y are the integral length scales in the wall normal and spanwise directions, respectively. N_{F_y} and N_{F_z} are the number of points for the wall normal and the spanwise correlation lengths, respectively. Δy and Δ are the grid spacing in normal and spanwise directions, respectively, while δ^* is the boundary layer displacement thickness. More details can be found in Touber and Sandham (2008).

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