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Nonlinear stress analysis of plates under thermo-mechanical loads

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Abstract. The aim of the present work is to carry out a simplified mathematical modelling for nonlinear stress analysis of plates under temperature changes and mechanical transverse loads. The material properties of the plate are proposed to be temperature-dependent. The geometrically nonlinear plate theory is employed to understand the stress distribution due to thermo-mechanical loads. A set of coupled nonlinear partial differential equations are solved using harmonic series expansion to find the static responses. Two boundary conditions are considered for simply supported plates, namely, movable edges and immovable edges. The accuracy of the results is checked by comparing with the output of other solution methods.

1. Introduction

In plate theories the small deformation assumption is no longer acceptable when the deflection magnitude is of the same order as the plate thickness. Thus, large deformation theory is introduced which includes the effect of membrane forces and coupling between the axial reactions and the transverse deformation of the plate. In response to the need for new structural materials, materially nonlinear theory was also developed by introducing nonlinear constitutive relations for stresses and strains [1-3]. Later on, both geometrically and materially nonlinear theories were employed to analyse plates with large deflections, but the most common approach was using approximate solutions for plates with *movable* simply supported edges. For movable (stress-free) boundary condition the supported edges are free to move whereas the out-of-plane displacement is fixed. However, for *immovable* boundary condition, equivalent axial reaction loads could be defined to prevent both in-plane and out-of-plane displacements along the edges [4]. To the best authors knowledge few attempts were made to find a simplified mathematical procedure for large deflection analysis of rectangular plates under thermo-mechanical loads [5, 6].

In this paper an analytical method is developed to present a robust nonlinear analysis for plates under thermo-mechanical loads. Geometric nonlinearity theory is used to establish the nonlinear governing partial differential equations. The plate properties are assumed to vary with temperature according to the Eurocode [7]. The accuracy of the present methodology is checked by comparing the results with some exiting results for large deflections of plates.

2. Mathematical formulations

The inverse strain-stress relations for plane stress including the thermal strains are

$$E\epsilon_{xx} = (F_{,yy} - \nu F_{,xx}) + \alpha E\Delta T, \quad E\epsilon_{yy} = (F_{,xx} - \nu F_{,yy}) + \alpha E\Delta T, \quad E\gamma_{xy} = -2(1 + \nu)F_{,xy} \quad (1)$$

where E , F , ν , α and ΔT are modulus of elasticity, stress function, Poisson's ratio, coefficient of thermal expansion and temperature change, respectively. The compatibility equation formed by strain components can be expressed by

$$\epsilon_{xx,yy} + \epsilon_{yy,xx} - \gamma_{xy,xy} = w_{,xy}^2 - w_{,xx} w_{,yy} \tag{2}$$

Substitution from relations (1) into (2) and then integration over the thickness leads to

$$h \nabla^4 F + N^T_{,xx} + N^T_{,yy} - Eh (w_{,xy}^2 - w_{,xx} w_{,yy}) = 0 \tag{3}$$

where h is the plate thickness and N^T is the thermal stress resultant defined by

$$N^T = E\alpha \int_{-h/2}^{h/2} \Delta T dz \tag{4}$$

The other nonlinear partial differential equation is given by

$$D \nabla^4 w - h (F_{,yy} w_{,xx} - 2F_{,xy} w_{,xy} + F_{,xx} w_{,yy}) + M^T_{,xx} + M^T_{,yy} - q(x, y) = 0 \tag{5}$$

where q is the applied load per unit area and M^T is the thermal moment resultants expressed by

$$M^T = \frac{E\alpha}{1-\nu} \int_{-h/2}^{h/2} \Delta T z dz \tag{6}$$

The following functions satisfy the two governing equations (3) and (5) for a plate with length a , width b and simply supported edges

$$(w, q, N^T, M^T) = \sum_{m=1}^N \sum_{n=1}^N (w_{mn}, q_{mn}, N^T_{mn}, M^T_{mn}) \sin(\alpha_m x) \sin(\gamma_n y) \tag{7}$$

$$F = -2P_x y^2 - 2P_y x^2 + \sum_{m=1}^N \sum_{n=1}^N F_{mn} \sin(\alpha_m x) \sin(\gamma_n y) \tag{8}$$

where

$$(N^T_{mn}, M^T_{mn}, q_{mn}) = \frac{4(-1 + (-1)^m)(-1 + (-1)^n)}{mn\pi^2} (N^T, M^T, q) \tag{9}$$

Here $\alpha_m = m\pi/a$ and $\gamma_n = n\pi/b$. Two different in-plane boundary conditions are considered for simply supported plate, SS1 where $w = w_{,xx} = 0$ and SS2 where $u = v = w = w_{,xx} = 0$. P_x and P_y are the tensile loads defined at $x = 0, a$ and $y = 0, b$, respectively, for SS2 case which can be obtained by the following relations. The purpose is that the plate edges should be restricted to move along the x and y directions. Then, the axial displacements can be expressed by

$$u = \int_0^a \left\{ \frac{1}{E} (F_{,yy} - \nu F_{,xx}) - \frac{1}{2} w_{,x}^2 + \alpha \Delta T \right\} dx \tag{10}$$

$$v = \int_0^b \left\{ \frac{1}{E} (F_{,xx} - \nu F_{,yy}) - \frac{1}{2} w_{,y}^2 + \alpha \Delta T \right\} dy \tag{11}$$

Substitution from relations (7) and (8) into the above relations gives

$$u = \frac{-4aP_x}{E} + \frac{4a\nu P_y}{E} - \frac{aF_{mn}[\nu\alpha_m^2 + \gamma_n^2] [-1 + (-1)^m] [-1 + (-1)^n]}{Emn\pi^2} + a\alpha\Delta T - \frac{a}{8}\alpha_m^2 w_{mn}^2 \tag{12}$$

$$v = \frac{-4bP_y}{E} + \frac{4b\nu P_x}{E} - \frac{bF_{mn}[\alpha_m^2 + \nu\gamma_n^2] [-1 + (-1)^m] [-1 + (-1)^n]}{Emn\pi^2} + b\alpha\Delta T - \frac{b}{8}\gamma_n^2 w_{mn}^2 \tag{13}$$

where for immovable edges $u = v = 0$ and the tensile loads will be expressed by

$$P_x = -\frac{\gamma_n^2 F_{mn} [-1 + (-1)^m] [-1 + (-1)^n]}{mn\pi^2} - \frac{Ew_{mn}^2}{8(1-\nu^2)}(\alpha_m^2 + \nu\gamma_n^2) + \frac{N^T}{h(1-\nu)} \quad (14)$$

$$P_y = -\frac{\alpha_m^2 F_{mn} [-1 + (-1)^m] [-1 + (-1)^n]}{mn\pi^2} - \frac{Ew_{mn}^2}{8(1-\nu^2)}(\nu\alpha_m^2 + \gamma_n^2) + \frac{N^T}{h(1-\nu)} \quad (15)$$

Substituting relations (7) and (8) into equations (3) and (5) leads to

$$\begin{aligned} & \left((\alpha_m^2 + \gamma_n^2)^2 F_{mn} h - (\alpha_m^2 + \gamma_n^2) N_{mn}^T \right) \sin(\alpha_m x) \sin(\gamma_n y) \\ & - Eh \left\{ (\alpha_m \gamma_n w_{mn} \cos(\alpha_m x) \cos(\gamma_n y)) (\alpha_r \gamma_s w_{rs} \cos(\alpha_r x) \cos(\gamma_s y)) \right. \\ & \left. - (\alpha_m^2 w_{mn} \sin(\alpha_m x) \sin(\gamma_n y)) (\gamma_s^2 w_{rs} \sin(\alpha_r x) \sin(\gamma_s y)) \right\} = 0 \end{aligned} \quad (16)$$

$$\begin{aligned} & \left(D (\alpha_m^2 + \gamma_n^2)^2 - 4hP_x \alpha_m^2 - 4hP_y \gamma_n^2 \right) w_{mn} \sin(\alpha_m x) \sin(\gamma_n y) \\ & - h \left\{ (\gamma_s^2 F_{rs} \sin(\alpha_r x) \sin(\gamma_s y)) (\alpha_m^2 w_{mn} \sin(\alpha_m x) \sin(\gamma_n y)) \right. \\ & - 2 (\alpha_r \gamma_s F_{rs} \cos(\alpha_r x) \cos(\gamma_s y)) (\alpha_m \gamma_n w_{mn} \cos(\alpha_m x) \cos(\gamma_n y)) \\ & \left. + (\alpha_r^2 F_{rs} \sin(\alpha_r x) \sin(\gamma_s y)) (\gamma_n^2 w_{mn} \sin(\alpha_m x) \sin(\gamma_n y)) \right\} \\ & - \left((\alpha_m^2 + \gamma_n^2) M_{mn}^T - q_{mn} \right) \sin(\alpha_m x) \sin(\gamma_n y) = 0 \end{aligned} \quad (17)$$

Using the expansion theorem we obtain

$$\begin{aligned} & (\alpha_m^2 + \gamma_n^2)^2 h F_{mn} - (\alpha_m^2 + \gamma_n^2) N_{mn}^T \\ & - 4Eh (\alpha_m \gamma_n \alpha_r \gamma_s w_{mn} w_{rs} \zeta_{mnr s} - \alpha_m^2 \gamma_s^2 w_{mn} w_{rs} \eta_{mnr s}) = 0 \end{aligned} \quad (18)$$

$$\begin{aligned} & D (\alpha_m^2 + \gamma_n^2)^2 w_{mn} - 4h (\alpha_m^2 P_x + \gamma_n^2 P_y) w_{mn} - (\alpha_m^2 + \gamma_n^2) M_{mn}^T - q_{mn} \\ & - 4h F_{rs} w_{mn} (\alpha_m^2 \gamma_s^2 \eta_{mnr s} - 2\alpha_m \gamma_n \alpha_r \gamma_s \zeta_{mnr s} + \gamma_n^2 \alpha_r^2 \eta_{mnr s}) = 0 \end{aligned} \quad (19)$$

where

$$\zeta_{mnr s} = \frac{(-m + 2m(-1)^{2m+r} - m(-1)^r)(-n + 2n(-1)^{2n+s} - n(-1)^s)}{\pi^2(4m^2 - r^2)(4n^2 - s^2)} \quad (20)$$

$$\begin{aligned} \eta_{mnr s} = & \frac{(-2m^2 + 2m^2(-1)^r - (-1)^r r^2 + r^2(-1)^{2m+r})}{rs\pi^2(4m^2 - r^2)(4n^2 - s^2)} \\ & + \frac{(-2n^2 + 2n^2(-1)^s - (-1)^s s^2 + s^2(-1)^{2n+s})}{rs\pi^2(4m^2 - r^2)(4n^2 - s^2)} \end{aligned} \quad (21)$$

For uncoupled term approximation ($r = m$ and $s = n$), the above equations will be reduced to a cubic nonlinear equation as follows

$$\begin{aligned} & \frac{32Eh\alpha_m^4 \gamma_n^4 H_{mn}^2}{(\alpha_m^2 + \gamma_n^2)^2} w_{mn}^3 + \left(D (\alpha_m^2 + \gamma_n^2)^2 - 4h(\alpha_m^2 P_x + \gamma_n^2 P_y) + \frac{8N_{mn}^T H_{mn} \alpha_m^2 \gamma_n^2}{(\alpha_m^2 + \gamma_n^2)} \right) w_{mn} \\ & - (\alpha_m^2 + \gamma_n^2) M_{mn}^T - q_{mn} = 0 \end{aligned} \quad (22)$$

where

$$H_{mn} = \frac{-1 + 2(-1)^m + 2(-1)^n - (-1)^{3m} - (-1)^{3n} - 3(-1)^{m+n} + (-1)^{3m+n} + (-1)^{m+3n}}{3mn\pi^2} \tag{23}$$

The membrane stresses may be obtained by the stress function F as follows

$$\sigma_{xx} = F_{,yy} = \sum_{m=1}^N \sum_{n=1}^N \left(-4P_x - \gamma_n^2 F_{mn} \sin(\alpha_m x) \sin(\gamma_n y) \right) \tag{24}$$

$$\sigma_{yy} = F_{,xx} = \sum_{m=1}^N \sum_{n=1}^N \left(-4P_y - \alpha_m^2 F_{mn} \sin(\alpha_m x) \sin(\gamma_n y) \right) \tag{25}$$

$$\tau_{xy} = -F_{,xy} = - \sum_{m=1}^N \sum_{n=1}^N \alpha_m \gamma_n F_{mn} \cos(\alpha_m x) \cos(\gamma_n y) \tag{26}$$

The extreme-fiber bending stresses are expressed by

$$\acute{\sigma}_{xx} = \sum_{m=1}^N \sum_{n=1}^N \frac{Eh(\alpha_m^2 + \nu\gamma_n^2)}{2(1 - \nu^2)} w_{mn} \sin(\alpha_m x) \sin(\gamma_n y) \tag{27}$$

$$\acute{\sigma}_{yy} = \sum_{m=1}^N \sum_{n=1}^N \frac{Eh(\gamma_n^2 + \nu\alpha_m^2)}{2(1 - \nu^2)} w_{mn} \sin(\alpha_m x) \sin(\gamma_n y) \tag{28}$$

$$\acute{\tau}_{xy} = - \sum_{m=1}^N \sum_{n=1}^N \frac{Eh\alpha_m \gamma_n}{2(1 + \nu)} w_{mn} \cos(\alpha_m x) \cos(\gamma_n y) \tag{29}$$

where the Einstein summation convention over repeated indices m, n, r , and s are used. For any assumed N , a set of nonlinear algebraic equations will be derived. Figure 1 describes this procedure for determining the transverse displacement of the plate.

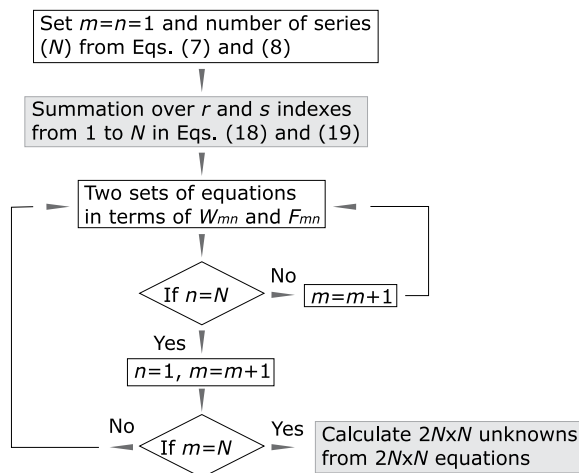


Figure 1. Flow chart of the solution programme.

3. Results and discussion

In this section, numerical examples for nonlinear analysis of plates under thermo-mechanical loads are presented to demonstrate the performance of the proposed method. The Poisson's ratio of the plate is assumed to be 0.3. Dimensionless parameters are considered as: centre deflection = w/h , load parameter = qa^4/Eh^4 , stress = $(\sigma_{xx}, \sigma'_{xx}, \tau'_{xy})a^2/Eh^2$.

The centre deflection for an isotropic square plate ($E = 7.8 \times 10^6$ psi, $a = b = 1$) subjected to uniformly distributed load is firstly compared in Figure 2 with the results of a finite element (FE) solution [8]. Numerical results are presented for $N = 1$ and $N = 3$ and for two specific simply supported boundary conditions, SS1 and SS2. Solutions for one term approximation ($N = 1$) match closely with FE solutions. As mentioned before, the material properties of the plate are assumed to be temperature-dependent. For this purpose Eurocode [7] suggests a trend for reduction of elasticity modulus of carbon steel with temperature which is plotted in Figure 3. Figure 4 shows the variation of extreme-fibre bending and membrane stresses in a square plate under uniformly distributed load and linear temperature changes ($\Delta T = 0.5 + T_z z$). The case associated with SS1 produces higher membrane and extreme-fibre bending stresses than SS2 case because of the existence of extension-bending coupling. It is observed that by increasing the value of load parameter, the plate is dominated by membrane stresses. For SS2 case, membrane stresses decrease due to the nature of the proposed stress function. The contour plots of extreme-fibre bending stresses for a square plate under thermo-mechanical loads are illustrated in Figure 5. The stress concentration for both patterns are similar to the finite element analyses.

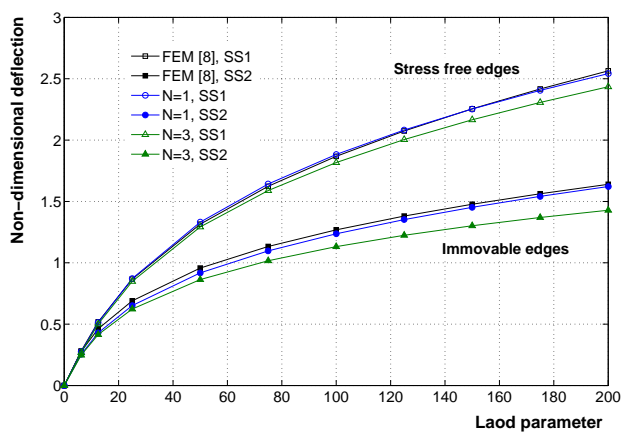


Figure 2. Convergence of the centre deflection for a square plate under uniformly distributed load.

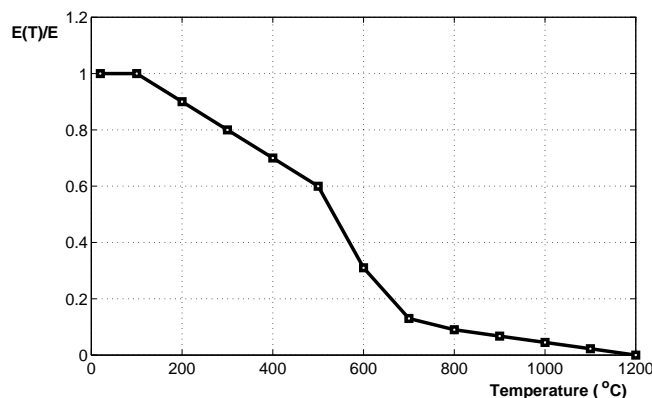


Figure 3. Reduction of elasticity modulus for carbon steel with temperature [7].

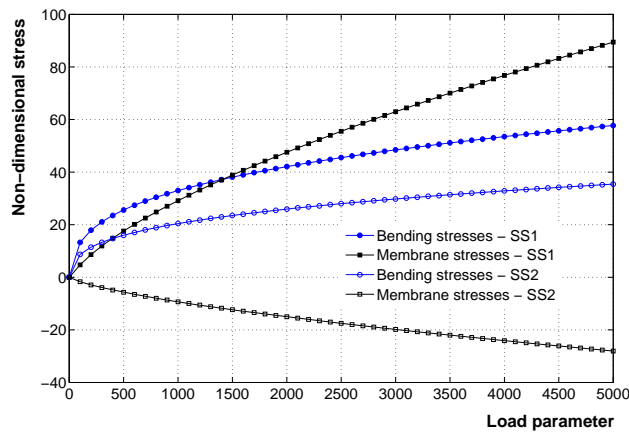


Figure 4. Bending and membrane stresses in a square plate under thermo-mechanical load ($N = 1, T_z = 200$).

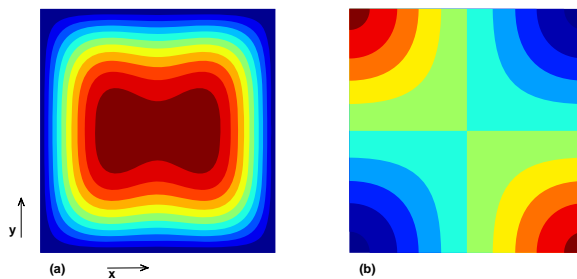


Figure 5. Contour plots of bending stresses for a square plate ($N = 3, T_z = 800$, load parameter=100): (a) $\sigma'_{xx} = 10.437$; (b) $\tau'_{xy} = -8.517$.

The present analysis has wide applications in structures under fire, particularly when elements of fire compartment boundaries such as wall panels are subjected to high temperatures and thermal gradients resulting in large displacements. Furthermore, evaluating new materials for aerospace applications could involve this kind of analysis.

4. Conclusions

A new mathematical formulation is developed for nonlinear stress analysis of plates with large displacements subjected to thermo-mechanical loads. The solution based on one term approximation was very close to those of other considered approaches whereas the solution of coupled terms provides more accurate results. The actual immovable edges can be simulated using an appropriate stress function. The results reveal that the effects of membrane action on large scale plates under thermo-mechanical loads are more than the effect of bending action.

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