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PII: S0307-904X(15)00551-X
DOI: 10.1016/j.apm.2015.09.017
Reference: APM 10716

To appear in: *Applied Mathematical Modelling*

Received date: 8 April 2013
Revised date: 6 March 2015
Accepted date: 23 September 2015

SOLITON SOLUTIONS TO COUPLED NONLINEAR EVOLUTION EQUATIONS MODELLING A THIRD HARMONIC RESONANCE IN THE THEORY OF CAPILLARY-GRAVITY WAVES

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Abstract Soliton solutions are sought to a pair of coupled nonlinear partial differential equations which model the interface of two stratified ideal fluids and which occurs when a fundamental mode and its third harmonic component induce a resonant interaction. These equations bear a resemblance to the standard coupled nonlinear Schrodinger equations, but they contain additional terms which makes their analysis quite different. There are two parameters in the problem: the ratio of the velocity of the fluids and of their densities. A large number of solutions is found and some important special cases are studied in detail.

Keywords. Nonlinear evolution equations, solitary waves, third harmonic resonance.

1. Introduction

Within the theory of capillary-gravity waves, it is manifest that the simplest examples of resonant interactions are those between the fundamental mode and its second and third harmonic. While the first scenario has been the subject of intense interest and even acquired its own appellation, that of Wilton ripples, the latter does not seem to have attracted anything like the same level of scrutiny. This is perhaps surprising for at least two reasons. The first is that these two types of resonance would seem to be the most likely to arise physically, both in nature and also in laboratory experiments. The second is that the mathematical equations which model these phenomena exhibit properties which make them quite different from those which model other resonant interactions. Despite the received nomenclature, one-two resonant capillary-gravity waves were the subject of at least two reports before Wilton's paper of 1915 [36]. One was the work of Bohr in 1906 [4] and the second [7] was due to Harrison in 1909. In view of the apparent obviousness of the development of the problem from one-two to one-three resonances, it seems scarcely believable that no researcher worked on this problem until 1961. However, this would appear to be the case for it was not until then that the paper of Pierson & Fife [33] considered these third harmonic resonances. However, even this treatment was fairly perfunctory. It was not until 1969 that Nayfeh [24,25] used the method of multiple scales to give a comprehensive description of third harmonic resonant surface waves. His results were extended to other physical systems by Verheest [35] and by Jones [11] who considered the stability of such waves and generalised later by Jones [17,19] who introduced the existence of an upper fluid.

A comprehensive investigation of resonant capillary-gravity waves by means of a formal power series expansion was carried out by Chen & Saffman [5]; this report included one-two and one-three resonances as well as perfectly general M-N ones. A more rigorous mathematical treatment of this problem was given later by Jones & Toland [20] who cast the problem as one in bifurcation theory. They employed functional analytic methods and confirmed most of Chen & Saffman's conclusions. The third harmonic interfacial problem was considered as one in bifurcation theory by Jones in [18]. All the reports above were devoted to travelling wave solutions of the third harmonic resonance. However the problem was re-considered by Dias & Bridges in [6] who showed by acknowledging the presence of O(2) symmetry that additional classes of solutions may exist. These new classes consist of standing waves, mixed waves and 'Z-waves'. In [10,32] Henderson and her co-researchers were able to create third harmonic resonant waves in a ripple tank by means of highly accurate experimental techniques. They showed that such waves were excited by a wavemaker of frequency 8.37 Hz and that they had a wavelength of 2.99 cm, thus confirming a prediction in [2].

In this paper we present a new set of soliton solutions to the pair of coupled nonlinear partial differential equations which model the evolution of a third harmonic resonant interaction. Soliton solutions to coupled pairs of evolution equations have been found by Wadeti [34], Newboult [29] and Ohta [30]. However because of the triad resonance, the set of equations studied here contains additional nonlinear terms which make their analysis quite different to these other cases. Also our equations contain two spatial co-ordinates while those mentioned above contain only one. As will become clear, for our solutions to exist it is essential that both coordinates are used. Finally although our physical motivation and setting is that of water-wave theory, nonlinear evolution equations such as those considered here may be employed to describe a wide range of physical phenomena. As examples we mention waves in a hot electron plasma [26-28]; in a circular jet [20]; monochromatic waves in an optical fibre with and without birefringence [23, 29, 34].

2. The equations of propagation

In this section we present a derivation of the system of partial differential equations which model the physical system. Since this procedure is well know we only sketch it, full details may be found in Jones [17], see also [11-14]. Our method is that of multiple scales and for other instances of the use of this method see [16, 24-28].Throughout the paper many of our calculation have been carried out using the computer algebra package maple.

Let us consider the three dimensional motion of two ideal fluids, one lying on top of the other. Throughout the following the subscript 1 refers to the lower fluid and 2 refers to the upper. The densities of the fluids are denoted by ρ_1 and ρ_2 and we shall assume the upper fluid floats on the lower so that $\rho_1 \geq \rho_2$, it will also prove useful to introduce the parameter $\rho = \frac{\rho_2}{\rho_1}$ which lies between zero and unity. We shall impose a three-dimensional Cartesian co-ordinate system, so that in the undisturbed state the equation for the flat interface is given by *z=0*. The fluids are supposed to have infinite vertical extent and the forces on the system are those of gravity *g*, which acts in the negative *z* direction and surface tension *S* which acts at the interface between the fluids. In the undisturbed state the fluids are in uniform horizontal laminar motion with velocity V_1 and V_2 in the direction parallel to the *x*-axis. We also introduce the

parameter V which is defined as $V = \frac{V_2}{V_1}$. This is depicted in Fig.1

Fig. 1

Our interest will centre on the small amplitude interfaces which are formed by the interaction between the fundamental mode and its third harmonic. We shall normalise and assume the wave-number of the fundamental is unity so that its wavelength is 2π. To this end we introduce the notation $E(n) = \exp in(x - \omega t)$ for any $n \geq 1$ where ω is the frequency. We also introduce the small positive parameter ε which acts as a measure of the interface wave steepness. The motion is regarded as irrotational and hence we are permitted to introduce velocity potentials $\varphi_j(x, y, z, t) \quad (j = 1,2)$ which satisfy Laplace's equation in each fluid. We also introduce a function $\zeta(x, y, t)$ so that the disturbed interface is given by $z = \zeta$.

We shall present the full nonlinear equations of motion shortly but first we shall use their linearisations to derive the dispersion relations. These are

$$\zeta_t - \varphi_{jz} + (V_j + \omega)\zeta_x = 0, \quad z = \zeta, \quad j = 1,2 \qquad (2.1a)$$

and

$$\rho\varphi_{2t} + \rho(\omega + V_2)\varphi_{2x} - \varphi_{1t} - (\omega + V_1)\varphi_{1x} + \frac{S}{\rho_1}(\zeta_{xx} + \zeta_{yy}) = 0, \ \ z = \zeta. \qquad (2.1b)$$

We seek solutions of the form $\varphi_1 = AE(n)e^{nz}$, $\varphi_2 = BE(n)e^{-nz}$, $\zeta = CE(n)$.

Substituting into (2.1a) yields $A = iV_1C$ and $B = -iV_2C$, while substituting into (2.1b) yields

$$\rho n V_2^2 + nV_1^2 - g(1-\rho) - \frac{n^2 S}{\rho_1} = 0. \qquad (2.2)$$

We wish this relationship to be satisfied for n=1,3. This means that

$$\rho V_2^2 + V_1^2 = g(1-\rho) + \frac{S}{\rho_1} \ \text{ and } \ 3(\rho V_2^2 + V_1^2) = g(1-\rho) + \frac{9S}{\rho_1} \qquad (2.3)$$

which in turn implies

$$\frac{S}{\rho_1} = \frac{V_1^2 + \rho V_2^2}{4} \ \text{ and } \ g(1-\rho) = \frac{3(V_1^2 + \rho V_2^2)}{4} \qquad (2.4)$$

and we shall henceforth assume that these relationships are satisfied.

The fully nonlinear boundary conditions at the interface then take the form:

$$\zeta_t - \varphi_{jz} + (V_j + \omega)\zeta_x + \varphi_{jx}\zeta_x + \varphi_{jy}\zeta_y = 0, \quad z = \zeta, \quad j = 1,2. \qquad (2.5)$$

and

$$\rho\varphi_{2t} + \rho(\omega + V_2)\varphi_{2x} - \varphi_{1t} - (\omega + V_1)\varphi_{1x} -$$
$$\frac{3}{4}(V_1^2 + V_2^2)\zeta + \frac{\rho}{2}(\varphi_{2x}^2 + \varphi_{2y}^2 + \varphi_{2z}^2) - \frac{1}{2}(\varphi_{1x}^2 + \varphi_{1y}^2 + \varphi_{1z}^2) +$$
$$\frac{(V_1^2 + V_2^2)(\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_x\zeta_y\zeta_{xy})}{4(1+\zeta_x^2+\zeta_y^2)^{3/2}} = 0, \ \ z = \zeta. \qquad (2.6)$$

The next step is to expand the expressions for the velocity potentials and interface profile as power series in ε. To facilitate this exercise we introduce the ‘slow space variables' $X = \varepsilon x, Y = \varepsilon y$ and the 'very slow time variable' $T = \varepsilon^2 t$. It might be thought that we should have also included a slow time scale, $T_1 = \varepsilon t$ say. However such terms drop out of the expressions during the calculations , see [16] eq (2.13a) or [19] eq (2.16) so we do not consider them.

Then the series expansion for φ_1 is, up to third order:

$$\varphi_1 = \varepsilon\left[iV_1C_1 + \varepsilon(A_1^{(2)} + zV_1C_{1X}) + \varepsilon^2(A_1^{(3)} - iz\,A_{1X}^{(2)} - \frac{izV_1}{2}C_{1YY} - \frac{iz^2V_1}{2}C_{1XX})\right]E(1)e^z +$$

$$\varepsilon\left[iV_1C_3 + \varepsilon(A_3^{(2)} + zV_1C_{3X}) + \varepsilon^2(A_3^{(3)} - iz\,A_{3X}^{(2)} - \frac{izV_1}{6}C_{3YY} - \frac{iz^2V_1}{2}C_{3XX})\right]E(3)e^{3z} +$$

$$\varepsilon^2 A(2)E(2)e^{2z} + \varepsilon^2 A(4)E(4)e^{4z} + \varepsilon^2 A(6)E(6)e^{6z} + cc, \qquad (2.7)$$

together with similar expressions for φ_2 and ζ.

In the expansions the coefficients are functions of the slow variables only and *c.c.* stands for complex conjugate. The derivation then proceeds by substituting into the boundary conditions and matching like terms.

At cubic order and using some standard scaling transformations, we arrive at the following pair of equations

$$iC_{1T} - p(1)C_{1XX} - 3C_{1YY} + u(1)|C_3|^2 C_1 + v(1)|C_1|^2 C_1 + 3w(1)C_3C_1^{*2} = 0, \qquad (\Im a)$$

$$iC_{3T} - p(3)C_{3XX} - \frac{5}{3}C_{3YY} + v(3)|C_3|^2 C_3 + u(1)|C_1|^2 C_3 + w(1)C_1^3 = 0. \qquad (\Im b)$$

The value of the coefficients *p(1)* etc may be found in the appendix.

These equations are the same as those derived in [18] apart from some elementary scaling simplifications. The system $(\Im)$ models up to cubic order the evolution of a capillary-gravity wavetrain at the interface of two fluids which is caused by the interaction between the fundamental mode and its third harmonic. There are two parameters in the problem: $V = \frac{V_2}{V_1}$ which is the ratio of the fluid velocities and $\rho = \frac{\rho_2}{\rho_1}$ which is the ratio of their densities. The parameter V can take any value, positive or negative, since we do not assume the fluids propagate in the same direction, while ρ must lie between zero and unity. The case *V=0* (ie V_2=0) corresponds to the case of resonant waves on the free surface of a single fluid with constant atmospheric pressure on the free surface. For a study of this see [11].

The system ($\Im$) bears a resemblance to that derived in [16] (see also [12,15]) which considered the general *M-N* interaction apart from the 1:2 and 1:3 resonances. The difference between the equations presented there and those here is that the terms involving $w(1)$ are absent. This means that the solutions are quite different. Soliton solutions of equations similar to those presented in [12,16] were found in [34,30]. However the equations considered there displayed a high degree of symmetry in that the coefficients *v(1)* and *v(3)* were assumed to be equal and *p(1)* and *p(3)* were assumed to be equal to unity. In addition those equations contained only one spatial derivative.

3. The soliton solutions

This section contains the main results of this paper. In the first subsection we derive a general soliton solution to the system ($\Im$). In the two subsequent subsections we look at some important particular cases in detail.

3.1 General solutions

Encouraged by [34] we shall seek solutions of ($\Im$) in the form

$$C_1 = A\, f(q) \exp[i\alpha\, X + i\beta\, Y + i\gamma\, T] \tag{3.1a}$$

$$C_3 = B\, f(q) \exp[3\,i\alpha\, X + 3i\beta Y + 3i\gamma\, T] \tag{3.1b}$$

where *f(q)=sech(q),* $q = a\,X + b\,Y - c\,T$ and $\alpha, \beta, \gamma, a, b, c, A, B$ are real constants.

Note that because of the terms $C_3 C_1^{*2}$ and C_1^3 in $(\Im)$ the complex argument of C_3 must be three times that of C_1. This constraint does not occur for the evolution equations solved in [30-34] and is a fundamental difference between those equations and $(\Im)$.

Substituting into $(\Im a)$ and taking the real part leads us to

$$\begin{aligned} &-\gamma\, f - p(1)a^2 f'' + p(1)\alpha^2 f - 3b^2 f'' + 3\beta^2 f + \\ &u(1)B^2 f^3 + v(1)A^2 f^3 + 3w(1)AB\, f^3 = 0. \end{aligned} \tag{3.2}$$

Then using the fact that $f'' = f - 2f^3$ and matching like powers of *f* presents us with

$$-\gamma - p(1)a^2 + p(1)\alpha^2 - 3b^2 + 3\beta^2 = 0,$$

(3.3a)

and

$$2p(1)a^2 + 6b^2 + u(1)B^2 + v(1)A^2 + 3w(1)AB = 0. \tag{3.3b}$$

Meanwhile, the imaginary part of $(\Im a)$ gives us

$$c + 2p(1)\alpha\, a + 6\beta\, b = 0\,. \tag{3.3c}$$

The corresponding equations for arising from $(\Im b)$ are

$$3\gamma - p(3)a^2 + 9p(3)\alpha^2 - \frac{5}{3}b^2 + 15\beta^2 = 0, \tag{3.4a}$$

$$2p(3)Ba^2 + \frac{10}{3}Bb^2 + v(3)B^3 + u(1)A^2B + w(1)A^3 = 0, \tag{3.4b}$$

$$c + 6p(3)\alpha\, a + 10\beta\, b = 0. \tag{3.4c}$$

If we now eliminate *c* between (3.3c) and (3.4c) we obtain

$$(p(1) - 3p(3))\alpha\, a = 2\beta\, b, \tag{3.5}$$

while eliminating γ between (3.3a) and (3.4a) presents us with

$$(p(3)-3p(1))a^2+3(p(1)-3p(3))\alpha^2-\frac{22}{3}b^2-6\beta^2=0. \tag{3.6}$$

This means that we have to solve the reduced system

$$(p(1)-3p(3))\alpha\, a=2\beta b, \tag{3.7a}$$

$$(p(3)-3p(1))a^2+3(p(1)-3p(3))\alpha^2-\frac{22}{3}b^2-6\beta^2=0, \tag{3.7b}$$

$$2p(3)Ba^2+\frac{10}{3}Bb^2+v(3)B^3+u(1)A^2B+w(1)A^3=0, \tag{3.7c}$$

$$2p(1)a^2+6b^2+u(1)B^2+v(1)A^2+3w(1)AB=0. \tag{3.7d}$$

It may now be noticed that for this method to work, it is most important that we consider three-dimensional waves and that the spatial dependence in the evolution equations is in both the *X* and *Y* directions. For if we were to consider waves that propagate in the *x*-directions only, so the *Y* terms were missing in the system $(\Im)$, then the right-hand side of (3.7a) would be zero. It is then easy to see from (3.7ab) that this would then force a and α to also be zero and hence no interesting solutions would arise. Of course there may be other classes of solutions which contain only one space variable. It should also be emphasised that our method only leads to the restricted (though large) class of sech solutions. More general solutions could be obtained by replacing *Af(q)* and *Bf(q)* in (3.1) with quite general functions *f(q)* and *g(q)*. This would then lead to a coupled system of nonlinear ordinary differential equations whose solutions in general will not be sech. It is probable that such a system would have to solved numerically, but this is an interesting avenue for further work.

The system (3.7) consists of four nonlinear algebraic equations for six unknowns: α,β,a,b,A and *B*. The most logical way now to proceed would seem to regard the leading wave amplitudes *A* and *B* as the independent variables. Then solving (3.7cd) for *a* and *b* leads to

$$a^2 = \frac{(9A^2B - 5B^3)u(1) - 5v(1)A^2B + (9A^3 - 15AB^2)w(1) + 9v(3)B^3}{2B(5p(1) - 9p(3))}, \qquad (3.8a)$$

$$b^2 = \frac{3p(3)B(u(1)B^2 + v(1)A^2 + 3w(1)A\ B) - 3p(1)(u(1)A^2B\ + v(3)B^3 + w(1)A^3)}{2B(5p(1) - 9p(3))}. \qquad (3.8b)$$

Then solving for α and β between (3.7ab) presents us with the following relatively simple expressions

$$\alpha^2 = \frac{2\{v(3)B^3 + (A^2B - 3B^3)u(1) + (A^3 - 9AB^2)w(1) - 3v(1)A^2B\}b^2}{3(p(1) - 3p(3))\{(B^3 - 3A^2B)u(1) - 3v(3)B^3 + v(1)A^2B + (3AB^2 - 3A^3)w(1)\}} \qquad (3.8c)$$

$$\beta^2 = \frac{(p(1) - 3p(3))\{v(3)B^3 + (A^2B - 3B^3)u(1) + (A^3 - 9AB^2)w(1) - 3v(1)A^2B\}a^2}{6\{(B^3 - 3A^2B)u(1) - 3v(3)B^3 + v(1)A^2B + (3AB^2 - 3A^3)w(1)\}}. \qquad (3.8d)$$

It may now be remarked that *A* and *B*, the amplitudes of the two leading harmonics, may be assigned any real values independently and corresponding solutions will almost certainly exist for a restricted, but fairly large, set of values of ρ and *V;* certainly this turns out to be the case for the examples we present in detail. This is in contrast to the solutions fond in [11-14] where the amplitudes of the leading harmonics could only assume a highly restricted set of values.

Owing to the large number of choices for *A* and *B*, we shall confine ourselves to four cases $B = \pm\frac{A}{3}$ and $B = \pm A$. We choose the first case we are dealing with a 1:3 resonance so that the amplitude of the third harmonic is one-third the amplitude of the fundamental. This approach is analogous to the one taken by Jones [14], Nayfeh [26] and McGoldrick [21,22] in the case of Wilton ripples. We consider the second case because then the evolution of the interface will be dependent on the interactions between the harmonics and not their relative magnitude.

Since $a, b\ \alpha, \beta$ must all be real, the signs of the various quantities on right-hand sides of (3.8) must be non-negative. We now note that the quantities $5p(1)-9p(3)$ and $p(1)-3p(3)$ (which occur in the denominators) are both always positive, this may be shown by an elementary completing the square calculation. Hence it is clear that β^2 is just a positive multiple of α^2 and its existence need not be considered further.

3.2 *The case* $B=\dfrac{A}{3}$.

We examine this case first because it is probably the simplest of the four examples which we consider.

Substituting into (3.8) now yields that

$$a^2=\frac{(76u(1)+9v(3)-45v(1)+198w(1))}{18(5p(1)-9p(3))}A^2, \tag{3.9a}$$

$$b^2=\frac{\{p(3)(u(1)+9v(1)+9w(1))-p(1)(9u(1)+v(3)+27w(1))\}}{6(5p(1)-9p(3))}A^2, \tag{3.9b}$$

$$\alpha^2=\frac{2\{27v(1)-v(3)-6u(1)\}b^2}{3(p(1)-3p(3))\{26u(1)+72w(1)+3v(3)-9v(1)\}}. \tag{3.9c}$$

It now turns out that a and α both exist by completing the square methods.

Hence the existence of a solution to the problem depends solely on the entity in the numerator of b^2 which equals

$$\frac{1}{60}\{98755\,\rho^2V^2-16077\rho-115961\rho^3V^4+45849-115961\rho V^2+98755\,\rho^2V^4$$
$$+186864\rho\,V-352288\rho^2V^3+186864\rho^3V^5-28203\rho^3V^6+45849\rho^4V^6\}. \tag{3.10}$$

The regions where (3.10) is positive and hence solutions of the system exist are depicted as shaded in Fig.2.

Fig. 2

3.3 The case $B = -\frac{A}{3}$.

The only difference between this and the case just discussed is that the sign of the coefficient of *w(1)* is changed in all the expressions in which it appears.

Calculations show that a^2 and α^2 both exist provided

$$0.466 < \rho V^2 < 2.15. \tag{3.11}$$

Hence for solutions for the problem to exist a necessary and sufficient condition is that in addition to (3.11), the expression occurring in the numerator of b^2 be positive. The regions of existence are shown in Fig. 3. As can be seen, when the velocity ratio V is negative, the region of existence is very narrow and the value of the density ratio ρ must be very small, less than 0.1 in fact. However, when V is positive, the region of existence is larger and more-or-less coincides with that defined by

$0.466 < \rho V^2 < 2.15$.

Fig. 3

3.4 The case B=A.

Here we have

$$a^2 = \frac{(4u(1) + 9v(3) - 5v(1) - 6w(1))}{2(5p(1) - 9p(3))} A^2, \tag{3.12a}$$

$$b^2 = \frac{3\{p(3)(u(1) + v(1) + 3w(1)) - p(1)(u(1) + v(3) + w(1))\}}{2(5p(1) - 9p(3))} A^2, \tag{3.12b}$$

$$\alpha^2 = \frac{2\{v(3) - 2u(1) - 8w(1) - 3v(1)\}b^2}{3(p(1) - 3p(3))\{v(1) - 2u(1) - 3v(3)\}}. \tag{3.12c}$$

Ccalculations similar to the previous ones now lead to the conclusion that both

a and α exists if and only if

either $0.190 < \rho V^2 < 0.532$ or $1.88 < \rho V^2 < 5.26$. (3.13)

For solutions to exist the numerator of b^2 must in addition be positive.

These regions of existence are as shaded in Fig. 4. As in the previous case, when the velocity ratio V is negative, the region of existence is very small and the value of the density ratio ρ must also be very small. For positive values of V there are two regions of existence.

Fig. 4

3.5 The case $B = -A$.

As before, when we evaluate the quantities *a,b* and α we find they are the same as in the previous case, except that the sign in front of *w(1)* is always changed.

Similar calculations now reveal that a and α exist if and only if

$$0.0667 < \rho V^2 < 0.0963 \quad \text{or} \quad 10.4 < \rho V^2 < 15.0. \tag{3.14}$$

The final condition for solutions to exist is the expression for b^2 positive, however a calculation shows the only places where this is negative lie outside the regions defined by (3.14) and hence the condition for existence is precisely that given by (3.14). These conclusions are depicted in Fig.5. Note that in this case the criterion for existence depends solely on the parameter ρV^2.

Fig. 5

4. Discussion and conclusions

The coupled nonlinear partial differential equations which model a third harmonic resonant interaction in two space dimensions and one time dimension have been studied. This is an important problem because this, together with second harmonic resonant interaction, is the one most likely to appear in nature and is the one easiest to reproduce in the laboratory. However despite its importance, this seems to be a little studied and oddly neglected problem. These equations exhibit mild symmetry conditions and are generalisations of the standard coupled nonlinear Schrodinger equations. However they contain certain additional terms which makes their analysis somewhat different. These terms impose a relationship between the wavenumbers and phases of the two expressions which form the solution. These equations have been shown to be integrable and explicit $sech^2$ soliton type solutions have been found. The existence of the solutions is dependent on the values of two parameters: ρ the density ratio and V the velocity ratio. It is shown that solutions exist for a large selection of values of the parameters.

The solutions which have been found are dependent on two spatial dimensions, there do not appear to be any analogous solutions in one spatial dimension only. This is in contrast to the case of the standard coupled nonlinear Schrodinger equations, see [].

Of course there may be other types of solution which are functions of *X* or *Y* only. There are a number of directions ways in which this research may be extended. Firstly, it would be interesting to see if we could relax the *ansatz* in (3.1) where the Soliton part of each solution is taken to be the same function *f* . If *f* is replaced by *g* in (3.1b), then this would lead to a system of coupled ordinary differential equations, the general solution of which would almost certainly have to be computed numerically. Another avenue of further study would be to see if the equations have any other solutions such as those corresponding to the Ma soliton or the peregrine soliton.

Appendix

We here present the expressions for the coefficients occurring in the system $(\Im)$.

We have

$$p(1) = \frac{\{\rho - 3 - 8\rho V + \rho(1-3\rho)V^2\}}{4}, \qquad (A1)$$

$$p(3) = \frac{\{\rho - 11 - 16\rho V \ + \rho(1-11\rho)V^2\}}{12}, \qquad (A2)$$

$$u(1) = 69 - 118\rho V^2 + 69\rho^2 V^4, \qquad (A3)$$

$$v(1) = \frac{\{77 - 102\rho V^2 + 77\rho^2 V^4\}}{2}, \qquad (A4)$$

$$w(1) = \frac{\{123 - 266\rho V^2 + 123\rho^2 V^4\}}{2}, \qquad (A5)$$

$$v(3) = \frac{\{5346\rho V^2 - 783 - 783\rho^2 V^4\}}{10}. \qquad (A6)$$

Note that only $p(1)$ and $p(3)$ depend on ρ and V explicitly, the other quantities are functions of the single entity ρV^2 only. It might be thought that conservation of energy considerations would suggest some form of symmetry in the coefficients of $(\Im a, \Im b)$. If we consider the analogous coefficients presented in [16] which dealt with the general *M-N* resonant case (although the 1:3 resonant case was specifically excluded in that report), we see that there is indeed a symmetry in *M* and *N*. However the transparency of this is lost for any particular numerical values. Note however that the coefficients of $|C_3|^2 C_1$ and $|C_1|^2 C_3$ are equal and that the coefficient of

$C_3 C_1^{*2}$ is three times that of C_1^3.

Acknowledgement The author thanks the referee for a careful reading of the manuscript and many helpful suggestions.

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Captions for Figures

Fig. 1 The flow configuration.

Fig. 2 The case $B = \frac{A}{3}$. The regions where (3.14) is positive and hence solutions exist is shaded.

Fig. 3a The case $B = -\frac{A}{3}$ and V negative. The bold lines show ρV^2 = 0.466 and 2.15. The regions where (3.18) is positive is shaded. Only the lower shaded region corresponds to an area of existence.

Fig. 3b The case $B = -\frac{A}{3}$ and V positive. The bold lines show ρV^2 = 0.466 and 2.15. The region where (3.18) is positive and hence solutions exist is shaded. Notice it just fits between the lines.

Fig. 4a The case $B = A$ and V negative. The dotted lines show ρV^2 =0.190, 0.532, 1.88, 5.26.The bold lines show the zeros of (3.25) which is positive in the region enclosed by them. Solutions exist in the shaded region.

Fig. 4b The case $B = A$ and V positive. The dotted lines show ρV^2 =0.190, 0.532, 1.88, 5.26.The bold lines show the zeros of (3.25) which is positive in the region between them. Solutions exist in the shaded regions.

Fig. 5 The case $B = -A$. The dotted lines show ρV^2 =0.0667, 0.0963, 10.4, 15.0. Solutions exist in the shaded regions. The bold lines show the zeros of (3.27) which is negative between them although this is irrelevant for our purposes.

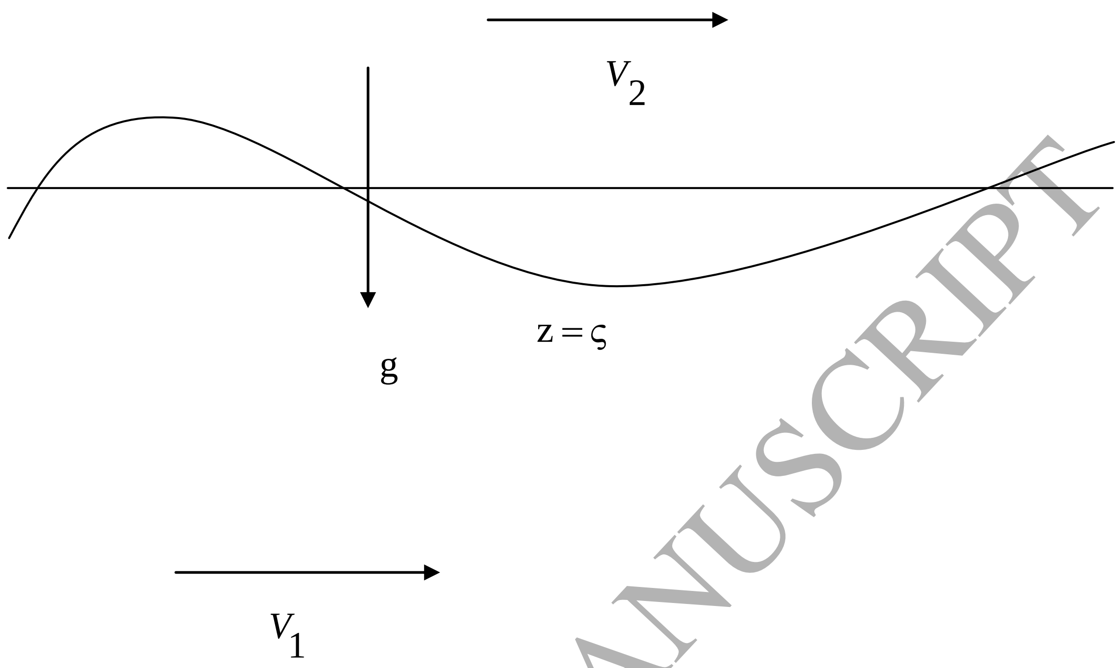

Fig 1

Fig 2

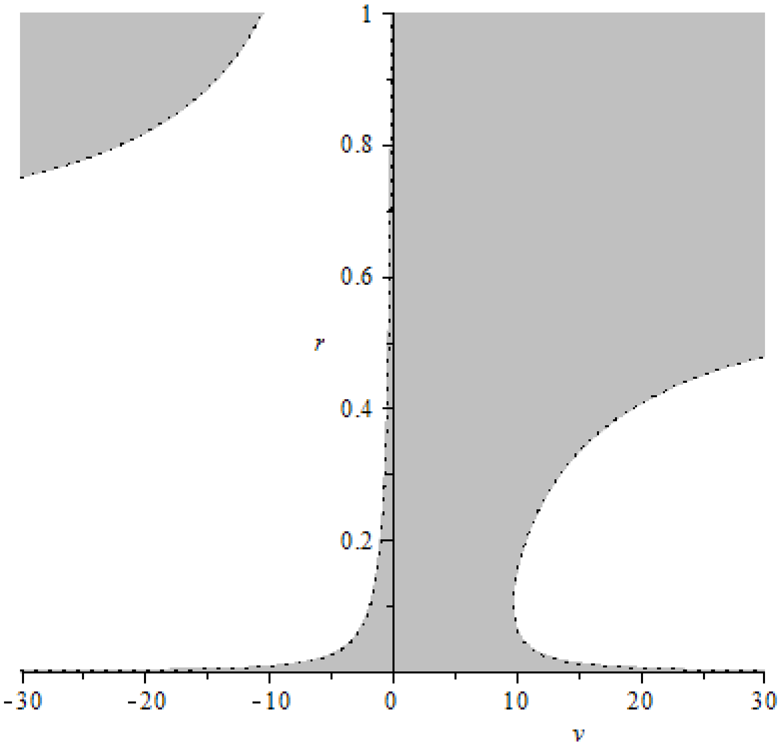

Fig 2

Fig 3a

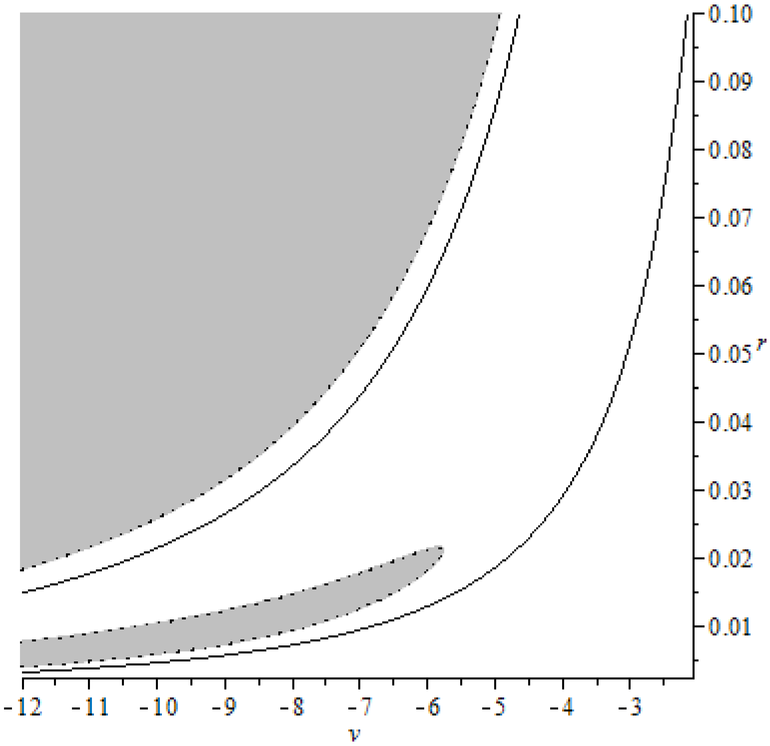

Fig 3a

Fig 3b

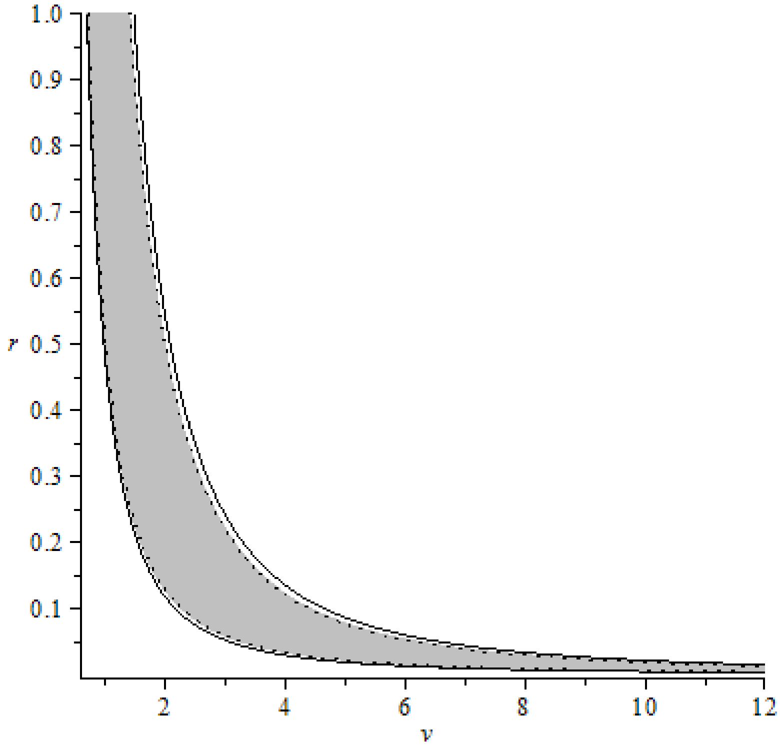

Fig 3b

Fig 4a

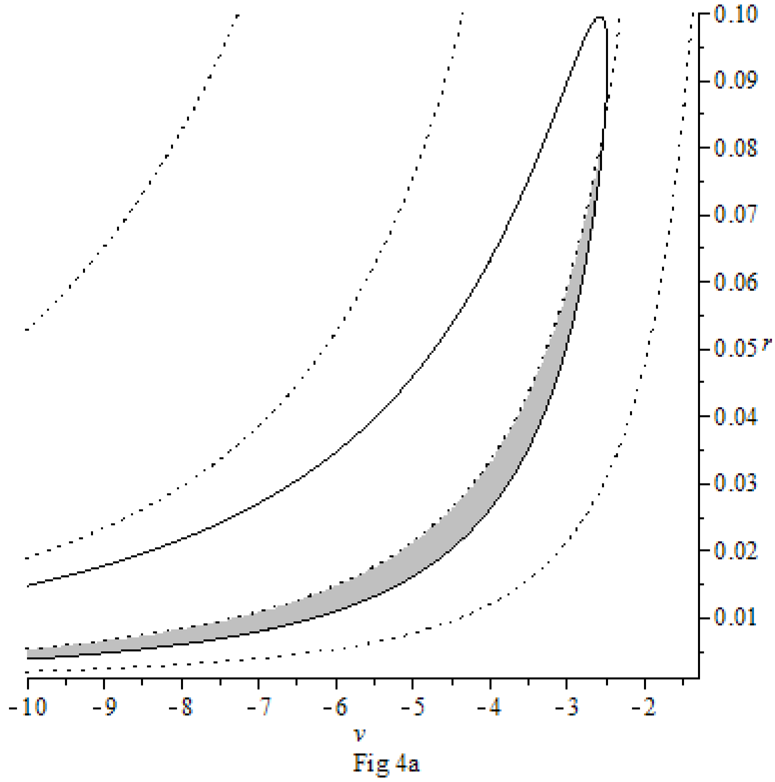
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Fig 4a

Fig 4b

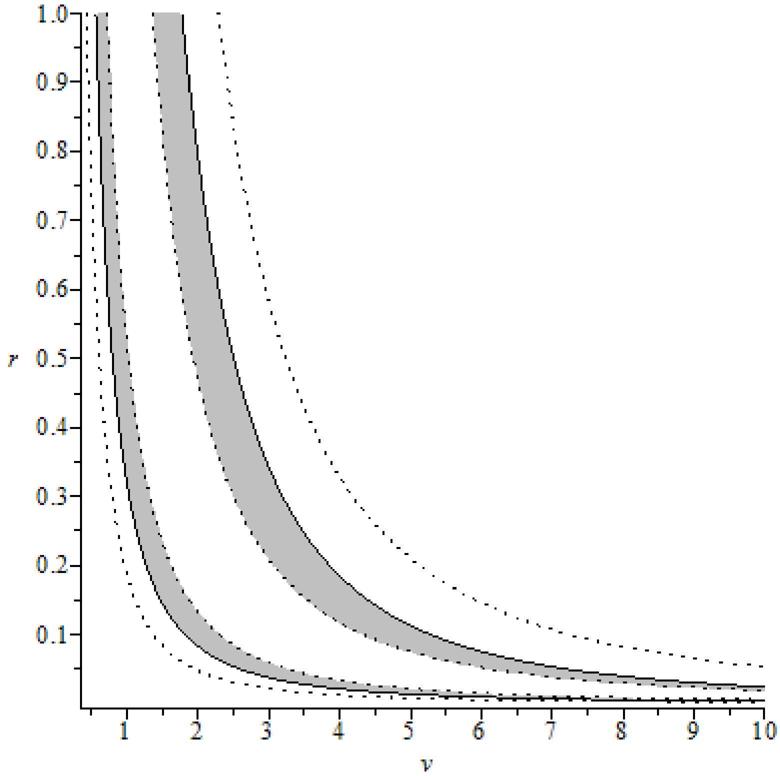
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Fig 5

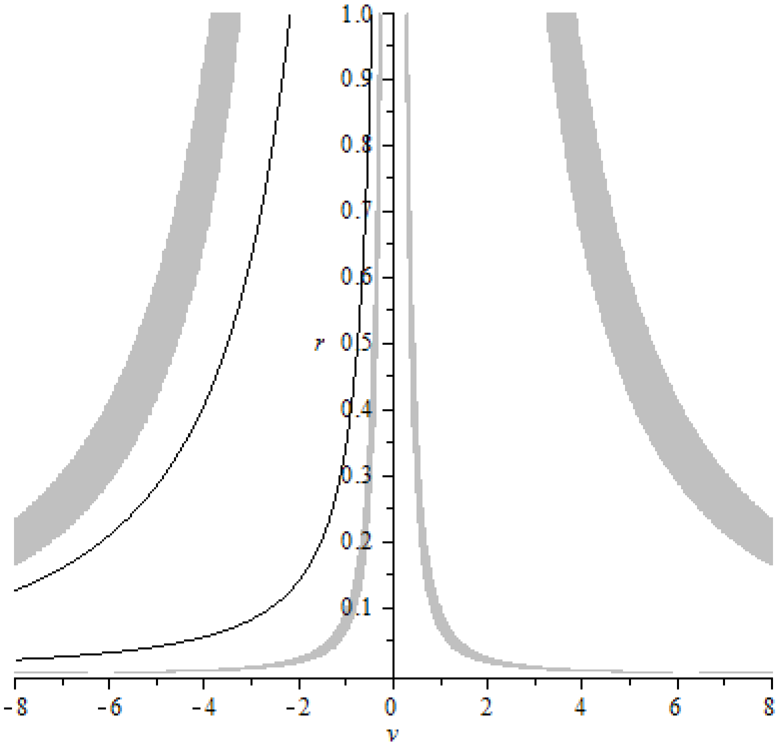

Fig 5