Innovation, Intellectual Property Rights, Imitation, and Income Distribution: Mathematical appendix

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1 Model solution and stability

This appendix contains the derivations of all results discussed in the main text. Equations without the 'A ' prefix refer to equations in the main text.

1.1 Derivation of equations (18) and (22)

In the absence of capital flows, the current account is balanced at any time and total expenditure $E^n = p^n n^n x^n$. Making use of the resource constraints total spending evolves according to

$$\frac{\dot{E}^n}{E^n} = \frac{\dot{p}^n}{p^n} - \frac{a\,\dot{g}}{(H^n - ag)} - \frac{\dot{a}_h}{a_h}. \tag{A.1}$$

From the monopoly pricing condition and the unit cost function (10), we can express the relative change in the price in terms of relative changes in the factor rewards w_h and w_l. The first order condition of profit maximization of a Northern firm and the factor market clearing condition gives us the relative wage rate prevailing in the North:

$$\omega \equiv \frac{w_l^n}{w_h^n} = \frac{1-b^n}{b^n}\left[\frac{H^n - ag}{L^n}\right]^{1-v^n}. \tag{A.2}$$

Differentiate the equation of the valuation of firms from the supply side (11) with respect to time and use (A.2), we obtain

$$\frac{\dot{p}^n}{p^n} = g + \frac{\dot{\vartheta}^n}{\vartheta^n} - \frac{(1-\phi_h^n)}{\sigma}\frac{a\,\dot{g}}{(H^n - ag)}. \tag{A.3}$$

Note, $\phi_h^n = (b^n)^{\sigma^n} w_h^{n1-\sigma^n}/[(b^n)^{\sigma^n} w_h^{n1-\sigma^n} + (1-b^n)^{\sigma^n} w_l^{n1-\sigma^n}]$, with $\sigma^n = 1/(1-\nu^n)$. Using (A.2), ϕ_h^n can also be expressed in terms of the exogenous factor supply.

The last part of the right hand side of (A.1) is obtained as follows. Using Shephard's Lemma we calculate the optimal capital input per unit of output from the unit cost function (10), say a_h^n. Together with (A.2)

$$\frac{\dot{a}_h}{a_h} = -\frac{(1-\phi_h^n)a}{(H^n - ag)}\,\dot{g}\;. \tag{A.4}$$

Substituting (A.3) and (A.4) into (A.1) and the resulting expression into (17) yields the differential equation (18) as given in the main text.

Here we derive (22) of the main text. Multiply the resource constraints (15) and (16) with their corresponding factor rewards and combined with (3) yields

$$\frac{c^n}{c^s}\frac{n^n}{n^s}\left(\frac{p^n}{p^s}\right)^{-\varepsilon}=\frac{w_h^n}{w_h^s}\left[\frac{(H^n-ag)+\omega^n L^n}{H^s+\omega^s L^s}\right]. \tag{A.5}$$

Substitute (A.2) into (A.5) and recall the definition of ζ, we obtain an equation for the terms of trade:

$$\frac{p^n}{p^s}=\left[\left(\tfrac{\zeta}{1-\zeta}\right)\left(\tfrac{b^s}{b^n}\right)^2\frac{\left(b^s(H^s)^{\nu^s}+(1-b^s)(L^s)^{\nu^s}\right)^{1/\nu^s}}{\left(b^n(H^n-ag)^{\nu^n}+(1-b^n)(L^n)^{\nu^n}\right)^{1/\nu^n}}\right]^{1/\varepsilon}. \tag{A.6}$$

We require the unit cost to be relatively smaller in the South implying that (A.6) must be greater than $c^n/c^s\alpha$. Solving for the rate of innovation provides condition (22).

1.2 Stability of the model

We start by log-linearizing equations (15) and (16) given in the main text and here reproduced for convenience:

$$\begin{aligned}\dot{\zeta}=&\ g-(g+m)\zeta\\ \dot{g}=&\left(\frac{H^n-ag}{a(1-\phi_l^n\nu^n)}\right)\left[g+\rho+m-\left(\frac{1-\alpha}{\alpha}\right)\frac{(H^n-ag)}{a\phi_h^n}\frac{1}{\zeta}\right].\end{aligned}$$

The log-linearized system written in matrix form is:

$$\begin{bmatrix}\dot{\widetilde{\zeta}}\,(t)\\ \dot{\widetilde{g}}\,(t)\end{bmatrix}=J\begin{bmatrix}\widetilde{\zeta}(t)\\ \widetilde{g}(t)\end{bmatrix}+\begin{bmatrix}-m\\ \frac{(H^n-ag)m}{a(1-\phi_l^n\nu^n)g}\end{bmatrix}[\widetilde{m}(t)], \tag{A.7}$$

where

$$J=\begin{bmatrix}-(g+m) & m\\ (g+\rho+m)\frac{(H^n-ag)}{a(1-\phi_l^n\nu^n)g} & \frac{g+\rho+m}{(1-\phi_l^n\nu^n)}\left\{\phi_h^n(\nu^n+\frac{\alpha}{1-\alpha}\zeta)+(1-\nu^n)\right\}\end{bmatrix},$$

and the Jacobian is evaluated around the initial steady state and where $\widetilde{\zeta}(t)\equiv d\zeta(t)/\zeta$, $\widetilde{g}(t)\equiv dg(t)/g, \widetilde{m}\equiv dm(t)/m$, $\dot{\widetilde{\zeta}}\,(t)\equiv\dot{\zeta}\,(t)/\zeta$, $\dot{\widetilde{g}}\,(t)\equiv\dot{g}\,(t)/g$, (where we have used $d\,\dot{\zeta}=\dot{\zeta}$, and $d\,\dot{g}\equiv\dot{g}$). We define the 2 by 2 matrix of the endogenous variables by Δ and its elements by a_{ij}; the elements of the column vector of the exogenous term are denoted by δ_i, $i=m,g$. Let $\lambda_1>0$ and $-\lambda_2<0$ be the roots of the characteristic equation of Δ. Recall that an equilibrium is saddle point stable when the roots of the characteristic equation are of different sign; i.e. $\lambda_1 > 0$ and $\lambda_2 > 0$. The characteristic roots are

$$\lambda_1 = \frac{tr\Delta + \sqrt{(tr\Delta)^2 - 4|\Delta|}}{2}$$

$$\lambda_2 = \frac{tr\Delta - \sqrt{(tr\Delta)^2 - 4|\Delta|}}{2},$$

where λ_2 denotes the speed of adjustment of the economy, $\mid \Delta \mid = -\lambda_1\lambda_2$, and the $tr\Delta = \lambda_1 - \lambda_2$. Using (A.7), the determinant of Δ can be written in most general terms as

$$\mid \Delta \mid = -\frac{(g+m+\rho)}{(1-\phi_l^n \nu^n)} \left\{ (g+m) \left[\phi_h^n (\nu^n + \frac{\alpha}{1-\alpha}\zeta) + (1-\nu^n) \right] + m \frac{(H^n - ag)}{ag} \right\}.$$

To show that the equilibrium is saddle point stable we have to show that the determinant is negative. We proceed by showing that the determinant is negative for all values of $\nu^n \in (-\infty, 1]$.

Case 1: $\nu^n = 1$ $(\sigma = \infty)$

$$\mid \Delta \mid = -(g+m+\rho) \left\{ (g+m)(1 + \frac{\alpha}{1-\alpha}\zeta) + m \frac{(H^n - ag) + (1-b)L^n}{bag} \right\} < 0.$$

Case 2: $\nu^n = 0$ $(\sigma = 1)$

$$\mid \Delta \mid = -(g+m+\rho) \left\{ (g+m) \left[\frac{b}{1-b} \frac{\alpha}{1-\alpha}\zeta + 1 \right] + m \frac{(H^n - ag)}{ag} \right\} < 0.$$

Case 3: $\nu^n = -\infty$ $(\sigma = 0)$

$$\mid \Delta \mid = -(g+m+\rho) \left\{ (g+m)(\frac{\alpha}{1-\alpha}\zeta + 1) + m \frac{(H^n - ag)}{ag} \right\} < 0,$$

iff $L^n/(H^n - ag) > 1$, and where we used the fact that

$$\lim_{v^n \to -\infty} \phi_l^n \nu^n = 0.$$

Hence the roots alternate in sign, and with $\lambda_1 > 0$ and $\lambda_2 > 0$ the equilibrium is a saddle point. $\square$

1.3 Model solution

We solve (A.7) by using Laplace transforms as suggested by Judd (1982, 1999) and further developed by Bovenberg & Heijdra (2001) and Heijdra (1999). In general, the Laplace transform $\mathfrak{L}\{\widetilde{f}, s\}$ of a function $\widetilde{f}(t)$ is defined by

$$\mathfrak{L}\{\widetilde{f}, s\} = \int_0^\infty \exp(-st)\widetilde{f}(t)dt.$$

The Laplace transform can therefore be interpreted as the present value discounted at rate s. We use the following expression for the Laplace transform of the time derivative of a function $\dot{\widetilde{f}}(t)$:

$$\mathfrak{L}\{\dot{\widetilde{f}}, s\} = \int_0^\infty \exp(-st)\,\dot{\widetilde{f}}(t)dt = s\mathfrak{L}\{\widetilde{f}, s\} - \widetilde{f}(0).$$

Applying the Laplace transform to (A.7) yields

$$\begin{bmatrix} s\mathfrak{L}\{\widetilde{\zeta}, s\} - \widetilde{\zeta}(0) \\ s\mathfrak{L}\{\widetilde{g}, s\} - \widetilde{g}(0) \end{bmatrix} = \Delta \begin{bmatrix} \mathfrak{L}\{\widetilde{\zeta}, s\} \\ \mathfrak{L}\{\widetilde{g}, s\} \end{bmatrix} + \begin{bmatrix} \delta_m \\ \delta_g \end{bmatrix} \mathfrak{L}\{\widetilde{m}, s\} \tag{A.8}$$

which implies that

$$\Lambda(s) \begin{matrix} \mathfrak{L}\{\widetilde{\zeta}, s\} \\ \mathfrak{L}\{\widetilde{g}, s\} \end{matrix} = \begin{bmatrix} \widetilde{\zeta}(0) + \delta_m \mathfrak{L}\{\widetilde{m}, s\} \\ \widetilde{g}(0) + \delta_g \mathfrak{L}\{\widetilde{m}, s\} \end{bmatrix}, \tag{A.9}$$

where we define $\Lambda(s) \equiv sI - \Delta$, and $\mid \Lambda(s) \mid = (s - \lambda_1)(s + \lambda_2)$.

Note, the rate of innovation, g, is a non-predetermined variable while the number of varieties not yet imitated, ζ, is a predetermined variable and consequently is not allowed to jump, i.e. $\widetilde{\zeta}(0) = 0$. Hence, The only unknown in (A.9) is the size of the jump in the rate of innovation at time 0, $\widetilde{g}(0)$. To find the initial jump in the rate of innovation, we use the condition that $\mathfrak{L}\{\widetilde{\zeta}, s\}$ and $\mathfrak{L}\{\widetilde{g}, s\}$ are bounded for $s = \lambda_1$. This implies that the right hand side of (A.9) should be zero for $s = \lambda_1 > 0$. Premultiplying (A.9) by $adj\Lambda(\lambda_1)$, the adjoint matrix of $\Lambda(\lambda_1)$, gives:

$$adj\Lambda(\lambda_1) \begin{bmatrix} \delta_m \mathfrak{L}\{\widetilde{m}, \lambda_1\} \\ \widetilde{g}(0) + \delta_g \mathfrak{L}\{\widetilde{m}, \lambda_1\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.10}$$

The system of equations in (A.10) provides two equivalent conditions for the jump in $\widetilde{g}$ on impact. This follows from the fact that since λ_1 is an eigenvalue, the two equations are not independent. As a consequence we have a unique expression for $\widetilde{g}(0)$ expressed in two alternative ways:[1]

$$\widetilde{g}(0) = -\delta_g \mathfrak{L}\{\widetilde{m}, \lambda_1\} - \left(\frac{\lambda_1 - a_{22}}{a_{12}}\right) \delta_m \mathfrak{L}\{\widetilde{m}, \lambda_1\} \tag{A.11}$$

$$\widetilde{g}(0) = -\delta_g \mathfrak{L}\{\widetilde{m}, \lambda_1\} - \left(\frac{a_{21}}{\lambda_1 - a_{11}}\right) \delta_m \mathfrak{L}\{\widetilde{m}, \lambda_1\}. \tag{A.12}$$

[1] See, for example, Judd (1999) page 459.

We can use either expression to eliminate $\widetilde{g}(0)$ from (A.9) and derive the general perfect foresight solution of the model in terms of Laplace transforms. Premultiplying (A.9) by the inverse of $\Lambda(s)$ and making use of

$$\begin{aligned} (sI-\Delta)^{-1} &= adj\Lambda(s)/\mid\Lambda(s)\mid \\ adj\Lambda(s) &= adj(\lambda_i)+(s-\lambda_i)I \qquad \text{for } i=1,2 \end{aligned}$$

in combination with (A.10) yields:

$$\begin{aligned} (s+\lambda_2)\begin{bmatrix} \mathfrak{L}\{\widetilde{\zeta},s\} \\ \mathfrak{L}\{\widetilde{g},s\} \end{bmatrix} = & \; adj\Lambda(\lambda_1)\begin{bmatrix} \delta_m\left[\frac{\mathfrak{L}\{\widetilde{m},s\}-\mathfrak{L}\{\widetilde{m},\lambda_1\}}{(s-\lambda_1)}\right] \\ \delta_g\left[\frac{\mathfrak{L}\{\widetilde{m},s\}-\mathfrak{L}\{\widetilde{m},\lambda_1\}}{(s-\lambda_1)}\right] \end{bmatrix} \\ & +\begin{bmatrix} \delta_m\mathfrak{L}\{\widetilde{m},s\} \\ \widetilde{g}(0)+\delta_g\mathfrak{L}\{\widetilde{m},s\} \end{bmatrix}. \end{aligned} \tag{A.13}$$

The long-run effects of tighter IPRs, $\widetilde{m}(\infty)$, are obtained from (A.13) by applying the final-value theorem (Spiegel, 1965, p.20).

$$\begin{aligned} \widetilde{g}(\infty) \equiv \lim_{s\to 0} s\mathfrak{L}\{\widetilde{g},s\} &= \tfrac{\widetilde{m}(\infty)}{\lambda_1\lambda_2}\left[a_{11}\delta_g - a_{21}\delta_m\right] \\ \widetilde{\zeta}(\infty) \equiv \lim_{s\to 0} s\mathfrak{L}\{\widetilde{\zeta},s\} &= \tfrac{\widetilde{m}(\infty)}{\lambda_1\lambda_2}\left[a_{22}\delta_m - a_{12}\delta_g\right]. \end{aligned} \tag{A.14}$$

By doing the appropriate substitution for a_{ij} and δ_i from (A.7) yields equations (28) and (30) in the main text.

By applying the initial value theorem (see Spiegel, 1965, p. 20) we can determine the initial development of the time rate of change in the rate of innovation and of the share of Northern products not yet imitated due to the shock:

$$\begin{aligned} \dot{\widetilde{g}}(0) &= \lim_{s\to\infty} s\mathfrak{L}\{\dot{\widetilde{g}},s\} = \lim_{s\to\infty} s\left[s\mathfrak{L}\{\widetilde{g},s\}-\widetilde{g}(0)\right] \\ &= \delta_g\widetilde{m}(0) - a_{21}\delta_m\mathfrak{L}\{\widetilde{m},\lambda_1\} - (\lambda_1-a_{11})\delta_g\mathfrak{L}\{\widetilde{m},\lambda_1\} \\ &= \delta_g\widetilde{m}(0) - a_{21}\left[\tfrac{(\lambda_1-a_{11})}{a_{21}}\delta_g\mathfrak{L}\{\widetilde{m},\lambda_1\}+\delta_m\mathfrak{L}\{\widetilde{m},\lambda_1\}\right] \\ \dot{\widetilde{g}}(0) &= \delta_g\widetilde{m}(0)+(\lambda_1-a_{11})\widetilde{g}(0). \end{aligned}$$

$$\begin{aligned} \dot{\widetilde{\zeta}}(0) &= \lim_{s\to\infty} s\mathfrak{L}\{\dot{\widetilde{\zeta}},s\} = \lim_{s\to\infty} s^2\mathfrak{L}\{\widetilde{\zeta},s\} \\ &= \delta_m\widetilde{m}(0) - a_{12}\left[\tfrac{(\lambda_1-a_{22})}{a_{12}}\delta_m\mathfrak{L}\{\widetilde{m},\lambda_1\}+\delta_g\mathfrak{L}\{\widetilde{m},\lambda_1\}\right] \\ \dot{\widetilde{\zeta}}(0) &= \delta_m\widetilde{m}(0)+a_{12}\widetilde{g}(0). \end{aligned}$$

Making use of the appropriate substitutions for the Cobb-Douglas case results in the expressions used in the main text.

In order to calculate the transition paths for the rate of innovation and the number of varieties not yet imitated, the intertemporal path of $\widetilde{m}(t)$ has to be specified. Inspired by Bovenberg & Heijdra (2001), we consider the case in which tighter intellectual property rights can be either abruptly or gradually implemented. The discussion in the main text makes use of the following parametrization:

$$\widetilde{m}(t) = (e^{-\beta t} - 1)\kappa, \tag{A.15}$$

where $\beta, \kappa > 0$. The Laplace transform of $\widetilde{m}(t)$ is

$$\frac{\mathfrak{L}\{\widetilde{m}, s\} - \mathfrak{L}\{\widetilde{m}, \lambda_1\}}{(s - \lambda_1)} = \frac{\kappa}{s\lambda_1} - \frac{\kappa}{(s+\beta)(\beta+\lambda_1)}. \tag{A.16}$$

It is helpful to recognize that

$$\begin{aligned} \tfrac{1}{(s+\beta)(s+\lambda_2)} &= \tfrac{1}{(\lambda_2-\beta)}\left[\tfrac{1}{(s+\beta)} - \tfrac{1}{(s+\lambda_2)}\right] \\ \tfrac{1}{s(s+\lambda_2)} &= \tfrac{1}{\lambda_2}\left[\tfrac{1}{s} - \tfrac{1}{(s+\lambda_2)}\right]. \end{aligned} \tag{A.17}$$

By substituting (A.16) into (A.13) and inverting the Laplace transform we calculate the transition path for the number of varieties not yet imitated and the rate of innovation in the time domain as:

$$\begin{aligned} \begin{bmatrix} \mathfrak{L}\{\widetilde{\zeta}, s\} \\ \mathfrak{L}\{\widetilde{g}, s\} \end{bmatrix} = & \begin{bmatrix} 0 \\ \widetilde{g}(0) \end{bmatrix} \tfrac{1}{(s+\lambda_2)} + \tfrac{adj\Lambda(0)\kappa}{\lambda_1\lambda_2} \begin{bmatrix} \delta_m \\ \delta_g \end{bmatrix} \tfrac{\lambda_2}{(s+\lambda_2)s} \\ & - \tfrac{adj\Lambda(0)\kappa}{(\lambda_1+\beta)} \begin{bmatrix} \delta_m \\ \delta_g \end{bmatrix} \tfrac{1}{(s+\lambda_2)(s+\beta)} + \tfrac{\beta\kappa}{(\beta+\lambda_1)} I \begin{bmatrix} \delta_m \\ \delta_g \end{bmatrix} \tfrac{1}{(s+\lambda_2)(s+\beta)}, \end{aligned}$$

where I is the identity matrix. Inverting the Laplace transform gives

$$\begin{aligned} \begin{bmatrix} \widetilde{\zeta}(t) \\ \widetilde{g}(t) \end{bmatrix} = & \begin{bmatrix} 0 \\ \widetilde{g}(0) \end{bmatrix} [1 + A(\lambda_2, t)] - \begin{bmatrix} \widetilde{\zeta}(\infty) \\ \widetilde{g}(\infty) \end{bmatrix} A(\lambda_2, t) \\ & - \tfrac{\kappa}{(\beta+\lambda_1)} [adj\Lambda(0) - \beta I] \begin{bmatrix} \delta_m \\ \delta_g \end{bmatrix} T(\beta, \lambda_2, t). \end{aligned}$$

Making use of (A.14) we get alternatively

$$\begin{aligned}
\widetilde{\zeta}(t) = & \; -\left[\widetilde{\zeta}(\infty)\right] A(\lambda_2,t) + \tfrac{\kappa}{(\lambda_1+\beta)}\left\{(a_{22}+\beta)\delta_m - a_{12}\delta_g\right\} T(\beta,\lambda_2,t), \\
\widetilde{g}(t) = & \; \widetilde{g}(0)\left[1+A(\lambda_2,t)\right] - \widetilde{g}(\infty)A(\lambda_2,t) \\
& + \tfrac{\kappa}{(\lambda_1+\beta)}\left\{(a_{11}+\beta)\delta_g - a_{21}\delta_m\right\} T(\beta,\lambda_2,t),
\end{aligned} \tag{A.18}$$

where A represents a single adjustment term given by

$$A(\lambda_2,t) \equiv (e^{-\lambda_2 t} - 1)$$

and T denotes a single transition term given by

$$T = \begin{cases} \frac{e^{-\beta t} - e^{-\lambda_2 t}}{\lambda_2 - \beta} & \text{for } \beta \neq \lambda_2, \\ t e^{-\lambda_2 t} & \text{for } \beta = \lambda_2. \end{cases} \tag{A.19}$$

The properties of $A(\lambda_2,t)$ and $T(\beta,\lambda_2,t)$ are as follows.

Lemma 1 *Let*

$$A(\alpha_1,t) \equiv (e^{-\alpha_1 t} - 1),$$

with $\alpha_1 > 0$ *and the following properties:*

(i) $A(\alpha_1,t) < 0$ for $t \in (0,\infty)$,
(ii) $A(\alpha_1,0) = 0$ for $t = 0$,
(iii) $\lim\limits_{t\to\infty} [A(\alpha_1,t)] = -1$,
(iv) $dA(\alpha_1,t)/dt \leq 0$,
(v) $A(\alpha_1,t)$ converges towards a unit step function $u(t)$ for $\alpha_1 \to \infty$.

Proof. Property (i), (ii), and (iii) are derived by substitution. Property (iv) follows from $dA(\alpha_1,t)/dt = -\alpha_1 e^{-\alpha_1 t}$ and the assumption that $\alpha_1 > 0$. Property (v) follows comparing the Laplace transforms of $u(t)$ and $A(\alpha_1,t)$ for $\alpha_1 \to \infty$. The Laplace transform of $u(t)$ is $\mathfrak{L}\{u(t),s\} = -1/s$ and the Laplace transform of $A(\alpha_1,t)$ is $\mathfrak{L}\{A(\alpha_1,t),s\} = 1/(\alpha_1+s) - 1/s$. For $\alpha_1 \to \infty$ property (v) is satisfied. $\square$

The properties of $T(\beta,\lambda_2,t)$, the single transition function, are identical to those discussed in Bovenberg & Hijdra (2001), Lemma A.2.

The jump in the rate of innovation that occurs at impact is derived by using either (A.11) or (A.12) in combination with (A.15):

$$\widetilde{g}(0) = \frac{\beta}{(\beta+\lambda_1)\lambda_1}\left[\delta_g + \left(\frac{\lambda_1 - a_{22}}{a_{12}}\right)\delta_m\right]\kappa \tag{A.20}$$

$$\widetilde{g}(0) = \frac{\beta}{(\beta+\lambda_1)\lambda_1}\left[\delta_g + \left(\frac{a_{21}}{\lambda_1 - a_{11}}\right)\delta_m\right]\kappa \tag{A.21}$$

Making use of the definition of a_{ij}, δ_m, and δ_g, the equations in the main text are derived.

Lemma 2 $\lambda_1 > \rho$.

Proof. The definition of the a_{ij} coefficients and

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[a_{11} + a_{22} + C\right] \qquad \text{where} \\ C &= \left[(a_{11}+a_{22})^2 - 4\left(a_{11}a_{22} - a_{21}a_{12}\right)\right]^{1/2} \end{aligned}$$

imply

$$-\left(a_{11}a_{22} - a_{21}a_{12}\right) > \rho\left[\rho - (a_{11}+a_{22})\right].$$

The left hand side of the last inequality represents the determinant, while the second term in the bracket on the right hand side denotes the trace. We previously established that $|\ \Delta\ | < 0$ and $tr\Delta > 0$ for $\nu^n \in (-\infty, 1]$. By substituting the expressions for the a_{ij} coefficients from (A.7) into this inequality establishes that the left hand side is positive while the right hand side is negative. It follows that $\lambda_1 > \rho$.□

1.3.1 Proof that North's terms of trade improve

Log-linearize (A.6)

$$\varepsilon\left(\widetilde{\frac{p^n}{p^s}(t)}\right) = \frac{1}{(1-\zeta)}\widetilde{\zeta}(t) + \frac{\phi_h^n a g}{(H^n - ag)}\widetilde{g}(t)$$

and make use of (A.14) yields

$$\begin{aligned} \varepsilon\left(\widetilde{\frac{p^n}{p^s}(\infty)}\right) = {} & \tfrac{\kappa m}{\lambda_1\lambda_2(1-\phi_l^n\nu^n)(1-\zeta)} \\ & \times\left\{(g+\rho+m)\left(\phi_h^n(\nu^n + \tfrac{\alpha}{(1-\alpha)}[\zeta+1]) + (1-\nu^n)\right) - (1-\zeta)\phi_h^n\rho\right\}. \end{aligned}$$

It is tedious but straightforward to show that

$$\rho\left[\left(\phi_h^n(\nu^n+\tfrac{\alpha}{(1-\alpha)}[\zeta+1])+(1-\nu^n)\right)-(1-\zeta)\phi_h^n\right]>0,$$
$$\rho\left[(1-\phi_h^n)(1-\nu^n)+\tfrac{\alpha+\zeta}{1-\alpha}\phi_h\right]>0.$$

This proves that North's terms of trade improve in the long run when intellectual property rights are tightened. □

1.4 Proof of Proposition 1

In this part of the appendix we characterize the paths of the rate of innovation and the share of Northern varieties not yet imitated. For this purpose we derive expressions for the time rate of change in these variables. Using (A.13) and (A.15) in combination with (A.17), the time path for the rate of change in the share of Northern goods not yet imitated yields

$$\mathfrak{L}\{\dot{\widetilde{\zeta}},s\}\equiv s\mathfrak{L}\{\widetilde{\zeta},s\}=\tfrac{\beta\kappa}{\beta+\lambda_1}\left[\tfrac{\delta_m(\lambda_1-a_{22})+\delta_g a_{12}}{\lambda_1}-\tfrac{\delta_m(\beta+a_{22})+\delta_g a_{12}}{s+\beta}\right]\left[\tfrac{1}{s+\lambda_2}\right]\quad\text{(A.22)}$$

In a similar way, the time path for the rate of change in the rate of innovation is calculated to be

$$\begin{aligned}\mathfrak{L}\{\dot{\widetilde{g}},s\}&\equiv s\mathfrak{L}\{\widetilde{g},s\}-\widetilde{g}(0)=\\&=\tfrac{\beta\kappa}{(\beta+\lambda_1)\lambda_1}\left[\tfrac{a_{22}[a_{21}\delta_m+(\lambda_2+a_{22})\delta_g]}{\lambda_2+a_{22}}+\tfrac{[a_{21}\delta_m-(a_{11}+\beta)\delta_g]\lambda_1}{s+\beta}\right]\left[\tfrac{1}{s+\lambda_2}\right]\quad\text{(A.23)}\end{aligned}$$

where we use the relations $(\lambda_1-a_{22})/a_{12}=a_{21}/(\lambda_1-a_{11})=a_{21}/(\lambda_2+a_{22})$.

By means of (A.22) and (A.23) we are now in the position to prove Proposition 1. Non-monotonicity in the adjustment path of $\widetilde{g}$ and $\widetilde{\zeta}$ requires that the sign in these rate of changes do switch along the adjustment path.

Lemma 3 *Let the Laplace transform $F(s)$ corresponding to a function $f(t)$ be (Note, Lemma 3 is identical to Lemma A.6 in Bovenberg and Heijdra (2001), p. 10 and is here reproduced for convenience):*

$$F(s)\equiv\frac{A_1}{(s+\alpha_1)}+\frac{A_2}{(s+\alpha_1)(s+\alpha_2)},$$

where $\alpha_1>0$, $\alpha_2>0$ and $A_1>0$.

Then the sign of $f(t)$ is as follows:

(i) $A_2\geq 0\Rightarrow f(t)\geq 0$.

(ii) $A_2<0$ and $(\alpha_1-\alpha_2)(\frac{A_1}{A_2}+1)<0\Rightarrow f(t)\geq 0$ for $t\in[0,\infty)$.

(iii) $A_2<0$ and $(\alpha_1-\alpha_2)(\frac{A_1}{A_2}+1)>0\Rightarrow f(t)\geq 0$ for $t\in[0,\bar{t}]$ and $f(t)\leq 0$ for $t\in[\bar{t},\infty)$.

Proof. The inverse of $F(s)$ is (see Spiegel (1965) p. 5):

$$f(t) = A_1 e^{-\alpha_1 t} + A_2 T(\alpha_1, \alpha_2, t),$$

where

$$T = \begin{cases} \frac{e^{-\alpha_2 t} - e^{-\alpha_1 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2, \\ te^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Part (i) of the Lemma follows straightforward if $A_2 > 0$. We prove part (ii) and (iii) of Lemma 4 by deriving the condition under which $f(t)$ cuts the t-axis. The t, say $\bar{t}$, for which $f(t) = 0$ amounts to:

$$\bar{t} = \begin{cases} \frac{1}{\alpha_1 - \alpha_2} \ln \left[(\alpha_2 - \alpha_1) \frac{A_1}{A_2} + 1 \right] & \text{for } \alpha_1 \neq \alpha_2, \\ -\frac{A_1}{A_2} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

As a result, there exists a $\bar{t} < \infty$ iff $(\alpha_2 - \alpha_1)\frac{A_1}{A_2} + 1 > 0$. $\square$

We can now use Lemma 3 in order to prove Proposition 1. For Proposition 1(i) to hold we have to show that the inverted terms in square brackets in (A.23) are functions of time that change sign for $t \in [0, \infty)$. Lemma 3 shows that we can establish the non-monotonicity property by using the Laplace transform of $f(t)$ directly. The $F(s)$ function when applied to (A.23) has elements $\alpha_1 = \lambda_2$ and $\alpha_2 = \beta$ and

$$\begin{aligned} A_1 &= \frac{a_{22}[a_{21}\delta_m + (\lambda_2 + a_{22})\delta_g]}{\lambda_2 + a_{22}}, \\ A_2 &= [a_{21}\delta_m - (a_{11} + \beta)\delta_g]\lambda_1. \end{aligned}$$

By making the appropriate substitutions we find that the signs of

$$\begin{aligned} A_1 &= \frac{(H^n ag)m}{ag(1-\phi_l^n \nu^n)} \left[\frac{(g+m+\rho)}{(1-\phi_l^n \nu^n)} \phi_h^n \frac{\alpha}{1-\alpha} \zeta + \lambda_2 \right] \frac{a_{22}}{\lambda_2 + a_{22}} > 0, \\ A_2 &= -\frac{(H^n ag)m}{ag(1-\phi_l^n \nu^n)} (\rho + \beta)\lambda_1 < 0. \end{aligned}$$

Applying Lemma 3(iii) it is straightforward to show that the adjustment path of the rate of innovation is non-monotonic. If the policy shock is introduced abruptly, i.e. $\beta \to \infty$, then

$$F(s) = \frac{1}{(\lambda_2 + a_{22})(\lambda_2 + s)} [a_{22}a_{21}\delta_m - (\lambda_2 + a_{22})\delta_g(\lambda_1 - a_{22})] < 0$$

implying monotonic adjustment for the path of the rate of change in the rate of innovation. This proves part (i) of Proposition 1. Analogously, the relevant $F(s)$ function for (A.22) has elements $\alpha_1 = \lambda_2$ and $\alpha_2 = \beta$ and

$$\begin{aligned} A_1 &= \tfrac{\delta_m(\lambda_1 - a_{22}) + \delta_g a_{12}}{\lambda_1} \\ A_2 &= -\left[\delta_m(\beta + a_{22}) + \delta_g a_{12}\right]. \end{aligned}$$

Using the relations $(\lambda_1 - a_{22})/a_{12} = a_{21}/(\lambda_1 - a_{22})$ and Lemma 2, it is straightforward to show that

$$A_1 = \frac{(H^n - ag)}{a(1 - \phi_l^n \nu^n)g} \frac{a_{12}(\lambda_1 - \rho)}{\lambda_1(\lambda_1 + g + m)} > 0.$$

With regards to the sign of the coefficient A_2 we have to look at the limits. For $\nu^n = 1$,

$$A_2 = m \left[\beta + \tfrac{(b(H^n - ag) + (1-b)L^n)}{abg} \left((\frac{1-\alpha}{\alpha})(g+m) + (g-m) \right) \right] > 0.$$

For $\nu^n = 0$,

$$A_2 = m \left[\beta + \tfrac{(H^n - ag)}{ag} \left((\tfrac{1-\alpha}{\alpha}) \tfrac{(g+m)}{b} + \tfrac{(g-(1-b)m)}{(1-b)} \right) \right] > 0.$$

For $\nu^n = -\infty$,

$$A_2 = m \left[\beta + \tfrac{(H^n - ag)}{ag} \left((\tfrac{1-\alpha}{\alpha})(g+m) + (g-m) \right) \right] > 0.$$

Assuming continuity, Lemma 3(i) applies indicating that the adjustment path of the share of Northern goods not yet imitated must be monotonic. This proves part (ii) of Proposition 1. $\square$

2 Welfare analysis of tighter intellectual property rights

In this section, the welfare implications of tighter intellectual property rights are derived. The utility of an agent at time $t = 0$, the time the shock occurs, takes the form:

$$U_i(0) = \int_0^\infty e^{-\rho\tau} \log u_i(\tau) d\tau,$$

where instantaneous utility is characterized by a CES function:

$$u_i(t) = \left[\int_0^{n(t)} x(j)^\alpha dj \right]^{1/\alpha}. \qquad \text{(A.24)}$$

It is well known that given the CES preference structure life-time utility of household i in region k can be rewritten in terms of aggregate spending E and price index P:

$$U_i^k(0) = \int_0^\infty \log\left[\varphi_i^k E^k(\tau)/P(\tau)\right] \exp\left[-\rho\tau\right] d\tau, \tag{A.25}$$

where

$$\varphi_i^k \equiv \frac{w_l^k L_i^k + w_h^k H_i^k}{w_l^k L^k + w_h^k H^k} \qquad \text{with} \quad \sum\nolimits_{i=1}^{I^k} \varphi_i^k = 1$$

represents the factor income share of individual i in region k. Total spending of the North and the South consists of the following components. The North derives income from human capital, labor and profits. Profits are generated from firms producing in the North and are equal to $(1-\alpha)$ of revenues. Together, total spending in the North reads

$$\begin{aligned} E^n &= w_l^n L^n + w_h^n H^n + n^n \pi^n \\ &= \alpha^{-1}\left[\omega^n L^n + H^n - (1-\alpha)ag\right], \end{aligned} \tag{A.26}$$

where $(1-\alpha)ag/\alpha$ are savings. Combining the first order condition of profit maximization and the factor market equilibrium conditions the relative wage in the North, $\omega^n \equiv w_l^n / w_h^n$ is

$$\omega^n = \left(\frac{1-b}{b}\right)\left(\frac{H^n - ag}{L^n}\right)^{1-\nu^n}.$$

It is helpful to recognize that the share of human capital in the production of any variety is defined by

$$\phi_h^n = \frac{b(H^n - ag)^{\nu^n}}{b(H^n - ag)^{\nu^n} + (1-b)(L^n)^{\nu^n}}.$$

Similarly, total spending in the South consists of income derived from unskilled labor and human capital:

$$E^s = \omega^s L^s + H^s,$$

where the relative wage in the South is

$$\omega^s \equiv \frac{w_l^s}{w_h^s} = \left(\frac{1-b^s}{b^s}\right)\left(\frac{H^s}{L^s}\right)^{1-\nu^s}.$$

Note, we choose as numeraire $w_h^n = w_h^s = 1$. The price index P, in turn, is given by

$$P = n^{1/1-\varepsilon}\left[\zeta(p^n)^{1-\varepsilon} + (1-\zeta)(p^s)^{1-\varepsilon}\right]^{1/1-\varepsilon}. \tag{A.27}$$

Substitute (A.26) and (A.27) into (A.25) and utility of individual i located in the North reads

$$\begin{aligned}\log u_i^n = \; & \log\left[H^n + \omega^n L^n - (1-\alpha)ag\right] + \tfrac{1}{\varepsilon-1}\log n \\ & + \tfrac{1}{\varepsilon-1}\log\left[\zeta + (1-\zeta)\left(\tfrac{p^n}{p^s}\right)^{\varepsilon-1}\right] + \log\varphi_i^n + \\ & - \tfrac{1}{1-\sigma^n}\log\left[(b)^{\sigma^n} + (1-b)^{\sigma^n}(\omega^n)^{1-\sigma^n}\right].\end{aligned}$$

The first bracketed term on the right hand side denotes the effect of factor income and of savings on the current utility flow. The second expression captures the availability of varieties. The third term represents the production reallocation and terms of trade effect. The fourth expression represents the effect of the income share on the utility flow; the larger the share in total income the higher is utility from consumption. The last term denotes the change in the production cost. The first three terms were already introduced in different form by Helpman (1993). The inclusion of a second factor requires that the flow of utility accounts for the relative wage rate, the income share and production cost on the flow of consumption.

Similarly for an individual in the South:

$$\begin{aligned}\log u_i^s &= \log\varphi_i^s + \log\left[H^s + \omega^s L^s\right] + \frac{1}{\varepsilon-1}\log n - \log p^s \\ & + \frac{1}{\varepsilon-1}\log\left[\zeta\left(\tfrac{p^n}{p^s}\right)^{1-\varepsilon} + (1-\zeta)\right].\end{aligned}$$

To arrive at an expression for the change in the current flow of consumption we first differentiating totally the log of the instantaneous utility and substitute these terms subsequently into the utility function. The first step yields:

$$d\log u_i^n = a_g^n\widetilde{g} + a^n\widetilde{\zeta} + \tfrac{1}{\varepsilon-1}\widetilde{n} \tag{A.28}$$

$$d\log u_i^s = -a_g^s\widetilde{g} - a^s\widetilde{\zeta} + \tfrac{1}{\varepsilon-1}\widetilde{n}, \tag{A.29}$$

where the coefficients a_g^n and a^n and a_g^s and a^s are defined, respectively, as

$$\begin{aligned}a_g^n = \; & \left\{-(1-\nu^n)\left[\tfrac{\omega^n(h^n-h_i^n)}{(\omega^n+h^n)(\omega^n+h_i^n)} + \tfrac{\omega^n L^n}{H^n-ag}\phi_h^n - \phi_l^n\right]\right. \\ & \left. -\tfrac{1}{\varepsilon}\phi_h^n + \phi_h^n\tfrac{(1-\zeta)}{\varepsilon[\zeta\theta^{-\alpha}+(1-\zeta)]}\right\}\tfrac{ag}{H^n-ag},\end{aligned}$$

where $h^n \equiv H^n/L^n$ defines Northern's relative factor abundancy and $h^n \equiv H_i^n/L_i^n$ relative factor abundancy of a Northern individual. The variable a_g^n captures the endogenous change in variables caused by changes in the rate of innovation. The first term reflects the change in the factor income share; the second term denotes the change in aggregate expenditure via the change in factor prices; the third term denotes the change in the production cost of a variety; the fourth term reflects the change in savings; and the last term reflects the change in the terms of trade, keeping ζ and $(1-\zeta)$ constant. It can easily be shown that

$$\frac{\omega^n L^n}{H^n - ag}\phi_h^n = \phi_l^n.$$

As can be seen from the relative wages in both regions, only the Northern relative wage changes. This implies that only Northern unit cost of producing manufactures change which also causes the terms of trade to improve for the North. The higher unit cost enters negatively while the improved terms of trade enter positively in the welfare analysis. As it turns out, they are of equal magnitude and therefore cancel. As a result, a_g^n simplifies to

$$a_g^n = \left\{-(1-\nu^n)\left[\tfrac{\omega^n(h^n-h_i^n)}{(\omega^n+h^n)(\omega^n+h_i^n)}\right] - \frac{1}{\varepsilon}\phi_h^n + \phi_h^n \tfrac{(1-\zeta)}{\varepsilon[\zeta\theta^{-\alpha}+(1-\zeta)]}\right\}\tfrac{ag}{H^n-ag}. \quad \text{(A.30)}$$

In addition

$$a_\theta^n = \tfrac{\alpha\theta^\alpha}{(\varepsilon-1)[\zeta+(1-\zeta)\theta^\alpha]} > 0,$$
$$a_\zeta^n = -\tfrac{\zeta(\theta^\alpha-1)}{(\varepsilon-1)[\zeta+(1-\zeta)\theta^\alpha]} < 0,$$
$$a^n = a_\theta^n + a_\zeta^n$$

The corresponding expressions for the South are

$$a_g^s = \tfrac{\alpha\zeta}{(\varepsilon-1)[\zeta+(1-\zeta)\theta^\alpha]}\phi_h^n\tfrac{ag}{(H^n-ag)} > 0$$
$$a_\theta^s = \tfrac{\zeta}{(\varepsilon-1)[\zeta+(1-\zeta)\theta^\alpha]}\tfrac{\alpha}{(1-\zeta)} > 0,$$
$$a_\zeta^s = -\tfrac{\zeta(1-\theta^\alpha)}{(\varepsilon-1)[\zeta+(1-\zeta)\theta^\alpha]} > 0,$$
$$a^s = a_\theta^s + a_\zeta^s,$$

with $\theta > 1$ and $p^n/p^s = \theta^{1/\varepsilon}$, and where we used the relation $E^n = p^n x^n n^n = (H^n - ag)/\alpha\phi_h^n$. To derive an expression for the change in life-time utility due to tighter intellectual property rights of individual i in region k we totally differentiate (A.25) as of the time $t = 0$ and use (A.28) and (A.29):

$$\begin{aligned}
dU_i^n(0) &= a_g^n\int_0^\infty \widetilde{g}(\tau)e^{-\rho\tau}d\tau + a^n\int_0^\infty \widetilde{\zeta}(\tau)e^{-\rho\tau}d\tau + \tfrac{1}{\varepsilon-1}\int_0^\infty \widetilde{n}(\tau)e^{-\rho\tau}d\tau \\
dU_i^s(0) &= -a_g^s\int_0^\infty \widetilde{g}(\tau)e^{-\rho\tau}d\tau - a^s\int_0^\infty \widetilde{\zeta}(\tau)e^{-\rho\tau}d\tau + \tfrac{1}{\varepsilon-1}\int_0^\infty \widetilde{n}(\tau)e^{-\rho\tau}d\tau.
\end{aligned}$$

Note, all integrals are known except the last one. From the definition of $g \equiv \dot{n} / n$ it follows that the number of varieties available at time τ equals $\log n(\tau) = \log n(0) + \int_0^\tau g(\varsigma) d\varsigma$, which is used to evaluate the third integral as follows:

$$
\begin{aligned}
\int_0^\infty \widetilde{n}(\tau) e^{-\rho\tau} d\tau \qquad &= d\left\{\int_0^\infty \log n(0) e^{-\rho\tau} d\tau + \int_0^\infty \left[\int_0^t g(\varsigma) d\varsigma\right] e^{-\rho\tau} d\tau\right\} \\
&= d\left\{\int_0^\infty \log n(0) e^{-\rho\tau} d\tau + \int_0^\infty g(\varsigma) \left[\int_0^t e^{-\rho\tau} d\tau\right] d\varsigma\right\} \\
&= d\left\{\int_0^\infty \log n(0) e^{-\rho\tau} d\tau + \int_0^\infty \tfrac{g(\varsigma)}{\rho} e^{-\rho\varsigma} d\varsigma\right\} \\
\tfrac{1}{\varepsilon-1} \int_0^\infty \widetilde{n}(\tau) e^{-\rho\tau} d\tau \quad &= \tfrac{1}{\varepsilon-1} \tfrac{g}{\rho} \int_0^\infty \widetilde{g}(\varsigma) e^{-\rho\varsigma} d\varsigma.
\end{aligned}
$$

Note, when going from the second to the third line we reverse the order of integration. The change in life time utility is then given by

$$
\begin{aligned}
dU_i^n(0) &= a_g^n \int_0^\infty \widetilde{g}(\tau) e^{-\rho\tau} d\tau + a^n \int_0^\infty \widetilde{\zeta}(\tau) e^{-\rho\tau} d\tau + \tfrac{1}{\varepsilon-1} \tfrac{g}{\rho} \int_0^\infty \widetilde{g}(\tau) e^{-\rho\tau} d\tau \\
dU_i^s(0) &= -a_g^s \int_0^\infty \widetilde{g}(\tau) e^{-\rho\tau} d\tau - a^s \int_0^\infty \widetilde{\zeta}(\tau) e^{-\rho\tau} d\tau + \tfrac{1}{\varepsilon-1} \tfrac{g}{\rho} \int_0^\infty \widetilde{g}(\tau) e^{-\rho\tau} d\tau.
\end{aligned}
$$

This shows that the welfare impact depends on the Laplace transform of the induced change in the rate of innovation and the share of Northern goods not yet imitated evaluated at $s = \rho$:

$$
\begin{aligned}
dU_i^n(0) &= a_g^n \mathfrak{L}\{\widetilde{g}, \rho\} + a^n \mathfrak{L}\{\widetilde{\zeta}, \rho\} + \tfrac{1}{\varepsilon-1} \tfrac{g}{\rho} \mathfrak{L}\{\widetilde{g}, \rho\}, \\
dU_i^s(0) &= -a_g^s \mathfrak{L}\{\widetilde{g}, \rho\} - a^s \mathfrak{L}\{\widetilde{\zeta}, \rho\} + \tfrac{1}{\varepsilon-1} \tfrac{g}{\rho} \mathfrak{L}\{\widetilde{g}, \rho\}.
\end{aligned}
$$

The transition paths for $\widetilde{g}(t)$ and $\widetilde{\zeta}(t)$ are given by (A.18), so that $dU(0)$ can be expressed for an individual located in the North and in the South in most general terms respectively as

$$
\begin{aligned}
(\rho + \lambda_2) dU_i^n(0) &= \left[\tfrac{g}{\rho(\varepsilon-1)} + a_g^n\right] \widetilde{g}(0) + \left[\tfrac{g}{\rho(\varepsilon-1)} + a_g^n\right] \tfrac{\lambda_2}{\rho} \widetilde{g}(\infty) + \tfrac{\lambda_2}{\rho} a^n \widetilde{\zeta}(\infty) \\
&\quad + \tfrac{\kappa}{(\beta+\rho)(\lambda_1+\beta)} \left[a^n B_\zeta + \left(a_g^n + \tfrac{g}{\rho(\varepsilon-1)}\right) B_g\right]
\end{aligned} \tag{A.31}
$$

and

$$
\begin{aligned}
(\rho + \lambda_2) dU_i^s(0) &= \left[\tfrac{g}{\rho(\varepsilon-1)} - a_g^s\right] \widetilde{g}(0) + \left[\tfrac{g}{\rho(\varepsilon-1)} - a_g^s\right] \tfrac{\lambda_2}{\rho} \widetilde{g}(\infty) - \tfrac{\lambda_2}{\rho} a^s \widetilde{\zeta}(\infty) \\
&\quad - \tfrac{\kappa}{(\beta+\rho)(\lambda_1+\beta)} \left[a^s B_\zeta - \left(\tfrac{g}{\rho(\varepsilon-1)} - a_g^s\right) B_g\right],
\end{aligned} \tag{A.32}
$$

for $\beta \neq \lambda_2$ and B_g and B_ζ are defined respectively by

$$\begin{aligned} B_g \equiv & \quad (\{a_{11}+\beta\}\delta_g - a_{21}\delta_m) > 0 \\ B_\zeta \equiv & \quad (\{a_{22}+\beta\}\delta_m - a_{12}\delta_g) < 0. \end{aligned} \tag{A.33}$$

In the event that the policy shock is unanticipated, i.e. $\beta \rightarrow \infty$, the fourth expression in $dU_i^n(0)$ and $dU_i^s(0)$ approaches zero.

The signs of B_g and B_ζ are determined by making the appropriate substitutions. We calculate

$$\begin{aligned} B_g \equiv & \quad \tfrac{(H^n - ag)m}{a(1-\phi_l^n \nu^n)g}(\rho+\beta) > 0, \\ B_\zeta \equiv & \quad -\tfrac{m}{(1-\phi_l^n \nu^n)}\left\{(g+\rho+m)[\phi_h^n(\nu^n + \tfrac{\alpha}{1-\alpha}\zeta) + (1-\nu^n)] + \tfrac{(H^n-ag)m}{ag} + \beta\right\} < 0. \end{aligned}$$

Note, the sign of B_g and B_ζ is independent of the elasticity of substitution between factors of production. Expressions (A.31) and (A.32) provide the basis of our welfare evaluation of tighter IPRs in the South and we proceed by discussing each welfare expression in turn.

2.1 Welfare analysis for the South [Proof of Proposition 2]

Making use of a_g^s, a_θ^s and a_ζ^s coefficients, (A.11) and (A.14) the expression of the change in life-time utility due to a change in the regime of intellectual property rights for the South if the shock is announced amounts to

$$\begin{aligned} (\rho+\lambda_2)dU^s(0) = & \quad -\tfrac{\kappa(H^n-ag)m}{\rho(\varepsilon-1)\lambda_1 a(1-\phi_l^n\nu^n)}\left[\tfrac{\rho+g+m}{\lambda_1+g+m}\right] \\ & +a_g^s \tfrac{\kappa(H^n-ag)m}{\lambda_1 ag(1-\phi_l^n\nu^n)}\left[\tfrac{\rho+g+m}{\lambda_1+g+m}\right] - \tfrac{\zeta\lambda_2\alpha}{(\varepsilon-1)\rho[\zeta+(1-\zeta)\theta^\alpha](1-\zeta)}\widetilde{\zeta}(\infty) \\ & -\tfrac{\zeta\lambda_2(\theta^\alpha-1)}{(\varepsilon-1)\rho[\zeta+(1-\zeta)\theta^\alpha]}\widetilde{\zeta}(\infty). \end{aligned}$$

The first line shows the effect of product availability on welfare, which is negative. The initial increase in the amounts of varieties available is more than compensated by its subsequent drop. In case consumer value varieties per se their flow of utility decreases eventually. The second line reflects the change in Southern terms of trade holding constant the weights $\zeta(t)$ and $[1-\zeta(t)]$. The change in relative prices is brought about by changes in the rate of innovation and by the change in the number of goods produced in the North. The initial increase in the rate of innovation leads to a deterioration of South's terms of trade while the subsequent drop in the rate of innovation leads to an improvement. In present value terms the improvement in the terms of trade due to changes in the rate of innovation is positive. However, this is counteracted by higher Northern prices due to the increased demand for labor of the manufacturing sector generated by a higher fraction of goods produced there. This renders the total effect on welfare ambiguous. The following lemma, however, establishes that the overall welfare effect of the terms of trade is negative for the South. The third line denotes the effect of the changes in the interregional allocation of production on welfare, i.e. changes in $\zeta(t)$ holding relative prices constant. This last effect is negative since $\theta > 1$.

Lemma 4 *Let*

$$\tfrac{\kappa(H^n-ag)m}{\lambda_1 a(1-\phi_l^n\nu^n)}\left[\tfrac{\rho+g+m}{\lambda_1+g+m}\right]a_g^s-\tfrac{\zeta\lambda_2\alpha}{(\varepsilon-1)\rho[\zeta+(1-\zeta)\theta^\alpha](1-\zeta)}\widetilde{\zeta}(\infty)<0 \tag{A.34}$$

and it follows that $dU^s(0) < 0$.

Proof. Making use of the definition of a_g^s and $\zeta(\infty)$, (A.34) implies

$$\begin{aligned}\rho(1-\zeta)\phi_h &< (\lambda_1+g+m)\left[\phi_h\nu^n+\tfrac{\alpha}{1-\alpha}\phi_h+(1-\nu^n)\right]\Rightarrow\\ \lambda_1(1-\zeta)\phi_h &< (\lambda_1+g+m)\left[\phi_h\nu^n+\tfrac{\alpha}{1-\alpha}\phi_h+(1-\nu^n)\right]\end{aligned}$$

which, in turn, implies

$$\lambda_1 m[\phi_l(1-\nu^n)+\frac{\alpha}{1-\alpha}\phi_h]+(g\lambda_1+(g+m)^2)[\phi_h(\nu^n+\frac{\alpha}{1-\alpha})+(1-\nu^n]>0.$$

This completes the proof of Proposition 2(i).□

This result is in accordance with the one derived by Helpman (1993).

Matters are slightly more involved if the policy is introduced gradually, i.e. $\beta < \infty$, since the transition term now becomes relevant. Combining (A.32), (A.33), and making the appropriate substitutions the discounted flow of Southern utility when the shock is implemented gradually amounts to

$$\begin{aligned}(\rho+\lambda_2)dU_a^s(0) = {} & \frac{\beta}{(\lambda_1+\beta)}(\rho+\lambda_2)dU^s(0)\\ & -\frac{\beta}{(\lambda_1+\beta)}\frac{a^s}{\rho(\beta+\rho)}\left[\lambda_1\lambda_2\widetilde{\zeta}(\infty)-\kappa m\rho\right],\end{aligned} \tag{A.35}$$

where

$$\begin{aligned}a^s = {} & \frac{\zeta\alpha}{(\varepsilon-1)\rho\left[\zeta+(1-\zeta)\theta^\alpha\right](1-\zeta)}\\ & +\frac{\zeta\lambda_2(\theta^\alpha-1)}{(\varepsilon-1)\rho\left[\zeta+(1-\zeta)\theta^\alpha\right]}>0.\end{aligned}$$

Lemma 5

$$\left[\lambda_1\lambda_2\widetilde{\zeta}(\infty)-\kappa m\rho\right]>0.$$

Proof. Direct substitution for $\widetilde{\zeta}(\infty)$ and rearranging gives

$$(g+\rho+m)\left\{\phi_h(\varepsilon-1)\zeta\right\}+(g+m)(1-\phi_l\nu^n)+\frac{(H^n-ag)m}{ag}>0.$$

We next show that the welfare losses experienced by the South are smaller the more gradual IPRs are tightened.

Lemma 6

$$(\rho+\lambda_2)dU_a^s(0)>(\rho+\lambda_2)dU^s(0).$$

Proof.

$$\begin{aligned}(\rho+\lambda_2)\left[dU_a^s(0)-dU^s(0)\right]= & -\tfrac{\lambda_1}{\lambda_1+\beta}(\rho+\lambda_2)dU^s(0)\\ & -\tfrac{\beta}{(\lambda_1+\beta)\rho(\rho+\beta)}\left[\lambda_1\lambda_2\widetilde{\zeta}(\infty)-\kappa m\rho\right]\end{aligned}$$

which is, ceteris paribus, more likely to be positive the lower is β

$$-\lambda_1(\rho+\lambda_2)dU^s(0)>\frac{\beta}{\rho(\rho+\beta)}\left[\lambda_1\lambda_2\widetilde{\zeta}(\infty)-\kappa m\rho\right].$$

This completes the proof of Proposition 2(ii).□

2.2 Welfare analysis for the North [Proof of Proposition 3]

2.2.1 Change in aggregate Northern welfare

Next we turn to the change in aggregate Northern welfare, concentrating first on tje effect when the policy shock is introduced without announcement ($\beta\to\infty$). Note, we use aggregate expenditure E^n in our calculations implying that the income share effect is zero. In this case the change in life-time utility is given by

$$\begin{aligned}(\rho+\lambda_2)dU^n(0)= & -\tfrac{\kappa(H^n-ag)m}{\rho(\varepsilon-1)\lambda_1 a(1-\phi_l^n\nu^n)}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)\\ & +\tfrac{\kappa m\phi_h}{\lambda_1(1-\phi_l^n\nu^n)\varepsilon}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)\\ & -\tfrac{\kappa m}{\lambda_1(1-\phi_l^n\nu^n)}\tfrac{\phi_h(1-\zeta)\theta^\alpha}{\varepsilon[\zeta+(1-\zeta)\theta^\alpha]}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)+\tfrac{\lambda_2}{\rho}a_\theta^n\widetilde{\zeta}(\infty)\\ & -\tfrac{\lambda_2\zeta(\theta^\alpha-1)}{(\varepsilon-1)\rho[\zeta+(1-\zeta)\theta^\alpha]}\widetilde{\zeta}(\infty).\end{aligned}$$

The first line represents the effect on product availability; the second line reflects the change in savings pattern; the third line denotes the change in the terms of trade, holding constant weights $\zeta(t)$ and $(1-\zeta(t))$. The change in the terms of trade is caused by a shift in the rate of innovation and the change in the share of varieties not yet imitated. The last line denotes the interregional product allocation effect holding constant relative prices.

As shown by Helpman (1993), the negative product variety effect is larger than the change in the savings pattern and for rates of imitation close to zero, the negative production allocation effect more than compensates the positive terms of trade effect. We first show that the welfare loss on account of the variety effect is larger than the welfare gain on account of adjustments in savings and R&D investments rates, or

$$\tfrac{\kappa m(\rho+g+m)}{\lambda_1(1-\phi_l\nu^n)\varepsilon(\lambda_1+g+m)}\left[\phi_h^n-\tfrac{(H^n-ag)}{a\rho\alpha}\right]<0$$

if and only if

$$g<\frac{H^n}{a}-\rho\alpha\phi_h^n.$$

or

$$g<\left[\phi_h(\varepsilon-1)+1\right](1-\alpha)g_{(m=0)}+\alpha\frac{H^n}{a}$$

where $g_{(m=0)}$ is the steady state rate of innovation when the rate of imitation is zero. Since the rate of innovation increases with the rate of imitation, the last inequality holds.

Next we show that the welfare loss due to the reallocation of production more than compensates the welfare gain on account of improved terms of trade for small rates of imitation.

Lemma 7 *For m sufficiently small*

$$\tfrac{\phi_h^n(1-\zeta)\theta^\alpha}{\varepsilon[\zeta+(1-\zeta)\theta^\alpha]}\tfrac{ag}{(H^n-ag)}\left[\widetilde{g}(0)+\frac{\lambda_2}{\rho}\widetilde{g}(\infty)\right]+a_\theta^n\frac{\lambda_2}{\rho}\widetilde{\zeta}(\infty)+a_\zeta^n\frac{\lambda_2}{\rho}\widetilde{\zeta}(\infty)<0.$$

Proof. From the definition of a_g^n, a_θ^n, a_ζ^n, (A.11) and (A.14) it follows that

$$\tfrac{\kappa m(\rho+g+m)\theta^\alpha}{\lambda_1(1-\phi_l\nu^n)[\zeta+(1-\zeta)\theta^\alpha]\varepsilon}\Gamma_1,$$

where

$$\begin{aligned}\Gamma_1\equiv\ &\left\{-\tfrac{\phi_h^n(1-\zeta)}{(\lambda_1+g+m)}+\tfrac{1}{\rho}\left(1-\tfrac{\zeta(1-\theta^{-\alpha})}{\alpha}\right)\right.\\&\left.\times\left[\phi_h(\nu^n+\tfrac{\alpha}{1-\alpha}\zeta)+(1-\nu^n)\right]+(1-\zeta)\tfrac{\alpha}{1-\alpha}\phi_h\right\}.\end{aligned}$$

In order to determine the sign of Γ_1 we look at its limiting behavior when $m \to 0$, implying $\zeta \to 1$ and $\theta^\alpha \to \infty$. It follows that

$$-\frac{\kappa m(\rho+g+m)\theta^\alpha}{\lambda_1[\zeta+(1-\zeta)\theta^\alpha]\varepsilon}\frac{(1-\alpha)}{\alpha(1-\phi_l\nu^n)\rho}[(1-\phi_l^n\nu^n)+\frac{\alpha}{1-\alpha}\phi_h^n]<0.$$

This proves that the negative production allocation effect more than compensates the positive terms of trade effect for sufficiently small rates of imitation. As a result, Northern countries lose from tighter intellectual property rights for small rates of imitation. □

We next derive the welfare expression when the policy is introduced gradually, i.e. $\beta < \infty$. Combining (A.31), (A.33) and making the appropriate substitutions the discounted flow of utility for the North when the shock is announced is given by

$$\begin{aligned}(\rho+\lambda_2)dU_a^n(0) = &\frac{\beta}{\beta+\lambda_1}\{(\rho+\lambda_2)dU^n(0)\\ &+\frac{a_\theta^n+a_\zeta^n}{\rho(\beta+\rho)}\left[\lambda_1\lambda_2\widetilde{\zeta}(\infty)-\kappa m\rho\right]\Big\},\end{aligned} \tag{A.36}$$

where $a_\theta^n > 0$ and $a_\zeta^n < 0$, and the squared bracketed terms is positive by Lemma 5. To determine the sign of (A.36) we look at the limiting behavior of $a_\theta^n > 0$ and $a_\zeta^n < 0$ when $m \to 0$. It is easily shown that

$$a_\theta^n + a_\zeta^n < 0$$

for m sufficiently small. Following the line of argument used in Lemma 6, for m sufficiently small, the welfare losses for the North is smaller the lower is β.

This completes the proof of Proposition 3.□

2.2.2 Proof of Proposition 4

When the production function is of Leontief type, the change in Northern welfare for individual i is given by

$$\begin{aligned}(\rho+\lambda_2)dU_i^n(0) = &-\tfrac{\kappa(H^n-ag)m}{\rho(\varepsilon-1)\lambda_1 a(1-\phi_l^n\nu^n)}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)\\ &+\tfrac{\kappa m\phi_h^n}{\lambda_1(1-\phi_l^n\nu^n)\varepsilon}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)\\ &-\tfrac{\kappa m\phi_h^n}{\lambda_1(1-\phi_l^n\nu^n)}\tfrac{(1-\zeta)\theta^\alpha}{\varepsilon[\zeta+(1-\zeta)\theta^\alpha]}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)+\tfrac{\lambda_2}{\rho}a_\theta^n\widetilde{\zeta}(\infty)\\ &-\tfrac{\lambda_2\zeta(\theta^\alpha-1)}{(\varepsilon-1)\rho[\zeta+(1-\zeta)\theta^\alpha]}\widetilde{\zeta}(\infty).\\ &+\left\{\tfrac{(1-\nu^n)}{(1-\phi_l^n\nu^n)}\tfrac{\omega^n(h^n-h_i^n)}{(\omega^n+h^n)(\omega^n+h_i^n)}\tfrac{\kappa m}{\lambda_1}\left(\tfrac{\rho+g+m}{\lambda_1+g+m}\right)\right\}.\end{aligned}$$

For $\nu^n \rightarrow -\infty$, $(1-\nu^n)/(1-\phi_l^n \nu^n) \rightarrow 1$ iff $(H^n - ag)/L^n < 1$ and

$$\frac{\omega^n (h^n - h_i^n)}{(\omega^n + h^n)(\omega^n + h_i^n)} \rightarrow 0.$$

As a consequence, the last term approaches zero so that the sign of $(\rho + \lambda_2) dU^n(0)$ is determined by the previous effects. Applying L'Hôpital to the income share expression proves the second part of Proposition.

This completes the proof of Proposition 4.$\square$

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